

The explicit theta correspondence for reductive dual pairs ($Sp(p, q), O^*(4)$)

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Abstract

For every irreducible Harish-Chandra module of $O^*(4)$, we determine its theta lift to $Sp(p, q)$ in terms of the Langlands parameter, for all non-negative integers p and q . Our strategy is to determine the desired theta lifts by their infinitesimal characters and lowest K -types.

Keywords: Reductive Dual Pairs, Theta Correspondence, Langlands Classification

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1. Introduction

In representation theory of classical Lie groups, many interesting representations have been constructed via the local theta correspondence for reductive dual pairs (see [8] and [9]). To get the Langlands parameters of the representations thus obtained, and for other applications to automorphic forms, it is of great interest to understand the local theta correspondence as explicitly as possible.

For a real reductive group H with complexified Lie algebra \mathfrak{h} and a fixed maximal compact subgroup K_H , we denote by $\mathcal{R}(H)$ the set of all isomorphism classes of irreducible admissible (\mathfrak{h}, K_H) -modules. By Corollary 4.2.4 of [26], every admissible (\mathfrak{h}, K_H) -module is isomorphic to the subspace of K_H -finite vectors of an admissible representation of H . By this result, also let $\mathcal{R}(H)$ denote the admissible dual of H , i.e., the set of all infinitesimal equivalence classes of irreducible continuous admissible representation of H . By abuse of notation, for a class $\pi \in \mathcal{R}(H)$, also let π denote a Harish-Chandra module (resp. an admissible representation) in this class.

Let p, q be non-negative integers such that $p + q > 0$ and let n be a positive integer. We fix a maximal compact subgroup $K = Sp(p) \times Sp(q)$ of $G = Sp(p, q)$ and a maximal compact subgroup $K' = U(n)$ of $G' = O^*(2n)$. The group pair (G, G') can be embedded into a symplectic group $Sp = Sp(4n(p + q), \mathbb{R})$ such that G and G' are centralizer of each other (as algebraic groups) in this symplectic group. We call (G, G') a reductive dual pair in Sp . Furthermore, we fix a maximal compact subgroup U of Sp as in §3 such that $U \cap G = K$ and $U \cap G' = K'$.

Fix a nontrivial unitary character $\psi(t) = e^{2\pi it}$ of the additive group \mathbb{R} ($\mathbf{i} = \sqrt{-1} \in \mathbb{C}$). A unitary representation ω_ψ of the two-fold metaplectic cover \widetilde{Sp} of Sp was constructed by Shale

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[19] and Weil [27]. We call this representation the oscillator representation or the Weil representation. Let \widetilde{U} denote the inverse image of U in \widetilde{Sp} . We omit the subscript ψ and denote this Weil representation by $\omega_{p,q,n}$ and its Fock space (the subspace of \widetilde{U} -finite vectors of $\omega_{p,q,n}$) by $\mathcal{F}_{p,q,n}$. Since the two-fold metaplectic covers of both G and G' uniquely split, the Weil representation $\omega_{p,q,n}$ can be regarded as a representation of $Sp(p, q) \times O^*(2n)$. Let $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$ (resp. $\mathcal{R}(O^*(2n), \omega_{p,q,n})$) denote the set of elements in $\mathcal{R}(Sp(p, q))$ (resp. $\mathcal{R}(O^*(2n))$) which can be realized in the form $\mathcal{F}_{p,q,n}/\mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{F}_{p,q,n}$ is a $(\mathfrak{sp}(2(p+q), \mathbb{C}), Sp(p) \times Sp(q))$ -invariant (resp. $(\mathfrak{o}(2n, \mathbb{C}), U(n))$ -invariant) subspace. Similarly, we define the set $\mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$. Howe [3] proved that $\mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$ is the graph of a bijection between $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$ and $\mathcal{R}(O^*(2n), \omega_{p,q,n})$. This bijection is called theta correspondence for the reductive dual pair $(Sp(p, q), O^*(2n))$. For $\pi \in \mathcal{R}(Sp(p, q))$ and $\pi' \in \mathcal{R}(O^*(2n))$, π corresponds to π' in this bijection if and only if $\pi \otimes \pi' \in \mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$. If they correspond, we write $\theta_{p,q}(\pi') = \pi$ and $\theta_n(\pi) = \pi'$. Furthermore, we write $\theta_{p,q}(\pi') = 0$ (resp. $\theta_n(\pi) = 0$) if π' (resp. π) does not belong to $\mathcal{R}(O^*(2n), \omega_{p,q,n})$ (resp. $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$). Formally, we also define the theta correspondence for $(Sp(0, 0), O^*(2n))$ and $(Sp(p, q), O^*(0))$.

Fruitful achievements have been made in [1], [13], [14] and [15] on the explicit theta correspondence for reductive dual pairs of type I. For $(Sp(p, q), O^*(2n))$, the authors of [11] described the local theta correspondence explicitly for $p + q \leq n$ as well as some of the cases for $p + q > n$. In [12], the authors described the local theta correspondence for $(O(p, q), SL(2, \mathbb{R}))$, $(U(p, q), U(1, 1))$ and $(Sp(p, q), O^*(4))$ in terms of constituents of degenerate principal series. The explicit theta correspondence in terms of Langlands parameters remains unknown. R. Howe and J.-S. Li determined the theta correspondence for $(O(p, q), SL(2, \mathbb{R}))$ in terms of Langlands parameters in their unpublished preprint. In this paper, we will follow their strategy and determine the theta correspondence for $(Sp(p, q), O^*(4))$ in terms of Langlands parameters for all non-negative integers p and q .

The paper is arranged as follows: In §2.1 and 2.2, we introduce the Langlands parametrization for irreducible admissible representations of $Sp(p, q)$ and $O^*(2n)$. Every irreducible admissible representation of $Sp(p, q)$ (resp. $O^*(2n)$) is infinitesimal equivalent to a representation $\pi(r, \lambda, \Psi, \mu, \nu)$, where r is a non-negative integer, $\mu \in (\mathbb{Z}_{\geq 1})^r$, $\nu \in \mathbb{C}^r$, λ is the Harish-Chandra parameter for a limit of discrete series $\pi(\lambda, \Psi)$ of $Sp(p - r, q - r)$ (resp. $O^*(2(n - 2r))$) and Ψ is a system of positive roots of $Sp(p - r, q - r)$ (resp. $O^*(2(n - 2r))$) satisfying condition (A) of §2.1. In §2.3, we introduce the algorithm for calculating the lowest $Sp(p) \times Sp(q)$ -types (resp. $U(n)$ -types) of $\pi(r, \lambda, \Psi, \mu, \nu)$.

In §3, we review the correspondence between $Sp(p) \times Sp(q)$ -types and $U(n)$ -types in the joint harmonics $\mathcal{H}_{p,q,n}$ and the correspondence of infinitesimal characters. Moreover, we also introduce the following two results for reducing the problem.

Theorem A. (Lemma 3.32 of [11]) For $\pi' \in \mathcal{R}(O^*(2n))$, $(\theta_{p,q}(\pi'))^\vee = \theta_{q,p}((\pi')^\vee)$.

For a representation π of $Sp(p, q)$ (resp. $O^*(2n)$), we denote its contragredient representation by π^\vee .

Theorem B. (Theorem 1 of [16]) For $\pi' \in \mathcal{R}(O^*(2n))$, $\theta_{p,q}(\pi')$ is nonzero if $\min\{p, q\} \geq n$.

In §4, we introduce the induction principle and prove the following theorem.

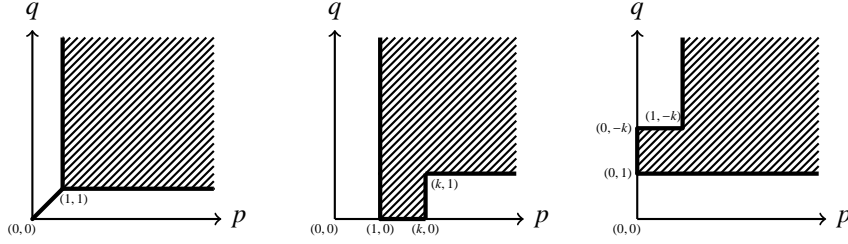
Theorem C. For $\pi' \in \mathcal{R}(O^*(2n))$, if $\theta_{p,q}(\pi') = \pi(r, \lambda, \Psi, \mu, \nu)$, then

$$\theta_{p+s, q+s}(\pi') = \pi(r + s, \lambda, \Psi, \mu^s, \nu^s),$$

where $\mu^s = (\mu, \underbrace{1, \dots, 1}_s)$ and

$$\nu^s = (\nu, 2p + 2q - 2n + 3, 2p + 2q - 2n + 7, \dots, 2p + 2q - 2n + 4s - 1).$$

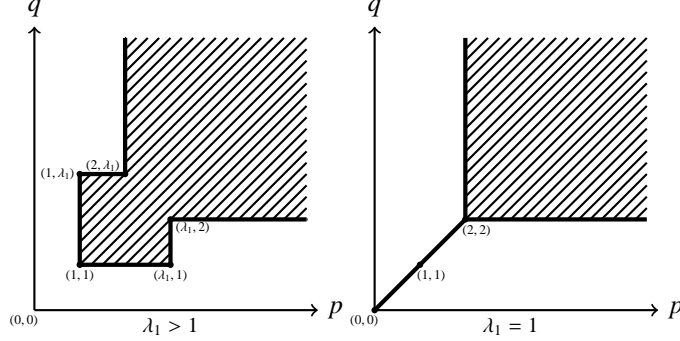
By Theorem A, Theorem B and Theorem C, we can reduce the problem to explicit description of $\theta_{p,q}$ for $p \geq q$ and $q \leq n$. In §5, we determine $\theta_{p,q}$ explicitly for reductive dual pairs $(Sp(p, q), O^*(2))$. This work is done by R. Howe and J.-S. Li in an unpublished preprint. For an integer k , let χ_k be the character $t \mapsto t^k$ of $O^*(2) \cong U(1)$. Since $O^*(2)$ is isomorphic to the compact group $U(1)$, every irreducible admissible representation of $O^*(2)$ is of the form χ_k . We define the occurrence set $O_k = \{(p, q) | \theta_{p,q}(\chi_k) \neq 0\}$. We give the occurrence pictures of χ_k for $k = 0$, $k > 0$ and $k < 0$ from left to right. The representation χ_k occurs in the theta correspondence for $(Sp(p, q), O^*(2))$ if and only if the integral point (p, q) lies in the shadow region or on the bold boundary line. The contragredient representation of χ_k is the representation χ_{-k} . By Lemma A, the occurrence picture of χ_k is just the reflection of the occurrence picture of χ_{-k} over the line $p = q$.



In §6, we determine $\theta_{p,q}$ explicitly for reductive dual pairs $(Sp(p, q), O^*(4))$. The following is a list of irreducible admissible representations (up to infinitesimal equivalence) of $O^*(4)$:

- (1) Irreducible principal series P_{λ_1, λ_2} with $\lambda_1, \lambda_2 \notin \mathbb{Z}$, $\text{Re}(\lambda_1 + \lambda_2) \geq 0$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (2) Limits of lowest weight discrete series D_{λ_1, λ_2} with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (3) Limits of highest weight discrete series $\overline{D}_{\lambda_1, \lambda_2}$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\leq 0}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (4) Finite dimensional representations F_{λ_1, λ_2} with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\geq 1}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$.

The representations in each case are determined by their infinitesimal characters (λ_1, λ_2) . We define the occurrence set for them as for those in $\mathcal{R}(O^*(2))$. The occurrence sets of all irreducible principle series are the same set $O_P = \{(p, q) | \min\{p, q\} \geq 1\}$. We denote the occurrence set of F_{λ_1, λ_2} by O_{λ_1, λ_2} , the occurrence set of D_{λ_1, λ_2} by $O_{\lambda_1, \lambda_2}^+$ and the occurrence set of $\overline{D}_{\lambda_1, \lambda_2}$ by $O_{\lambda_1, \lambda_2}^-$. By results in §6, we know that $O_{\lambda_1, \lambda_2}^+ = (1, 0) + O_{\lambda_2}$ and $O_{\lambda_1, \lambda_2}^- = (0, 1) + O_{\lambda_1}$. Then the occurrence picture of D_{λ_1, λ_2} is just the occurrence picture of χ_{λ_2} moved right by one unit and the occurrence picture of $\overline{D}_{\lambda_1, \lambda_2}$ is just the occurrence picture of χ_{λ_1} moved up by one unit. The following is a list of occurrence pictures of finite dimensional representations F_{λ_1, λ_2} .



Let $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5$ and Ψ_6 be the systems of positive roots mentioned in §5 and 6. For irreducible principal series, by Theorem A, Theorem C and the occurrence set O_p , we only need to calculate theta $(p, 1)$ -lifts for $p \geq 1$.

Theorem D. *The following is a list of theta $(p, 1)$ -lifts of irreducible principal series for $p \geq 1$:*

$$\theta_{p,1}(P_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

By the occurrence set $O_{\lambda_1, \lambda_2}^-$, we know that all theta $(p, 0)$ -lifts of limits of highest weight discrete series are zero. For limits of highest weight discrete series, we only need to calculate theta $(p, 1)$ -lifts and theta $(p, 2)$ -lifts. We list all theta $(p, 1)$ -lifts in the following theorem.

Theorem E. *The following is a list of theta $(p, 1)$ -lifts of limits of highest weight discrete series for $p \geq 1$:*

(1) *If $\lambda_1 \geq p$, then*

$$\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2}) = \pi((\lambda_1, p-1, p-2, \dots, 1; -\lambda_2), \Psi_2).$$

(2) *If $\lambda_1 < p$, then*

$$\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2}) = 0.$$

We list all theta $(p, 2)$ -lifts of limits of highest weight discrete series in the following theorem.

Theorem F. *The following is a list of theta $(p, 2)$ -lifts of limits of highest weight discrete series for $p \geq 2$:*

(1) *If $\lambda_1 \leq -p$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi((p, p-1, \dots, 1; -\lambda_2, -\lambda_1), \Psi_5).$$

(2) *If $\lambda_2 \leq -p+1 \leq \lambda_1 < p-1$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1; -\lambda_2), \Psi_2, p - \lambda_1, p + \lambda_1).$$

(3) *If $\lambda_2 > -p+1$ and $\lambda_1 < p-1$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (\nu_1, \nu_2)),$$

with $(\mu_1, \mu_2) = (p-1-\lambda_2, p-\lambda_1)$ and $(\nu_1, \nu_2) = (p-1+\lambda_2, p+\lambda_1)$.

(4) If $\lambda_1 \geq p - 1$, then $\theta_{p,2}(\overline{D}_{\lambda_1,\lambda_2})$ can be determined by Theorem C.

For finite dimensional irreducible representations, we have the following theorem.

Theorem G. *The following is a list of nonzero theta (p, q) -lifts of finite dimensional irreducible representations for $p \geq q$ and $q \leq 2$:*

- (1) $\theta_{0,0}(F_{1,0}) = \pi(0, \emptyset)$, where $\pi(0, \emptyset)$ is the trivial representation of the trivial group.
- (2) If $\lambda_1 \geq p$, then

$$\theta_{p,1}(F_{\lambda_1,\lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

- (3) If $\lambda_1 < p - 1$, then

$$\theta_{p,2}(F_{\lambda_1,\lambda_2}) = \pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (v_1, v_2)),$$

with $(\mu_1, \mu_2) = (p-1-\lambda_2, p-\lambda_1)$ and $(v_1, v_2) = (p-1+\lambda_2, p+\lambda_1)$.

- (4) If $\lambda_1 \geq p - 1$, then $\theta_{p,2}(F_{\lambda_1,\lambda_2})$ can be determined by Theorem C.

For limits of lowest weight discrete series, first we list all nonzero theta $(p, 0)$ -lifts.

Theorem H. *The following is a list of nonzero theta $(p, 0)$ -lifts of limits of lowest weight discrete series:*

- (1) $\theta_{1,0}(D_{\lambda_1,0}) = \pi((\lambda_1), \Psi_1)$.
- (2) If $p \geq 2$ and $\lambda_2 \geq p - 1$, then

$$\theta_{p,0}(D_{\lambda_1,\lambda_2}) = \pi((\lambda_1, \lambda_2, p-2, p-3, \dots, 1), \Psi_1).$$

Next we list all theta $(p, 1)$ -lifts for $p \geq 1$.

Theorem I. *The following is a list of theta $(p, 1)$ -lifts of limits of lowest weight discrete series for $p \geq 1$:*

- (1) If $\lambda_1 \leq p - 1$ and $\lambda_1 + \lambda_2 > 0$, then

$$\theta_{p,1}(D_{\lambda_1,\lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

- (2) If $\lambda_1 \leq p - 1$ and $\lambda_1 + \lambda_2 = 0$, then

$$\theta_{p,1}(D_{\lambda_1,-\lambda_1}) = \pi((p-1, \dots, \lambda_1+1, \lambda_1, \lambda_1, \lambda_1-1, \dots, 1; \lambda_1), \Psi)$$

with Ψ uniquely determined by condition (A) of §2.1.

- (3) If $\lambda_1 \geq p$ and $\lambda_2 \leq -p$, then

$$\theta_{p,1}(D_{\lambda_1,\lambda_2}) = \pi((\lambda_1, p-1, p-2, \dots, 1; -\lambda_2), \Psi_4).$$

- (4) If $p \geq 2$, $1-p < \lambda_2 \leq p-2$ and $\lambda_1 \geq p$, then

$$\theta_{p,1}(D_{\lambda_1,\lambda_2}) = \pi(1, (\lambda_1, p-2, p-3, \dots, 1), \Psi_1, p-1-\lambda_2, p-1+\lambda_2).$$

- (5) If $p \geq 2$, $\lambda_1 \geq p$ and $\lambda_2 = 1-p$, then

$$\theta_{p,1}(D_{\lambda_1,1-p}) = \pi((\lambda_1, p-1, p-2, \dots, 1; p-1), \Psi_4).$$

(6) If $p \geq 3$ and $\lambda_2 \geq p - 1$, then

$$\theta_{p,1}(D_{\lambda_1,\lambda_2}) = \pi(1, (\lambda_1, \lambda_2, p-3, p-4, \dots, 1), \Psi_1, 1, 2p-3).$$

(7) If $p = 2$ and $\lambda_2 \geq 1$, then

$$\theta_{2,1}(D_{\lambda_1,\lambda_2}) = \pi((\lambda_1, \lambda_2; 1), \Psi_3).$$

(8) If $p = 1$ and $\lambda_2 \geq 0$, then

$$\theta_{1,1}(D_{\lambda_1,\lambda_2}) = 0.$$

Finally, we calculate theta $(p, 2)$ -lifts of limits of lowest weight discrete series for $p \geq 2$. By Theorem C and Theorem I, we only need to calculate theta $(2, 2)$ -lifts of D_{λ_1,λ_2} for $\lambda_2 \geq 0$.

Theorem J. *The following is a list of theta $(2, 2)$ -lifts of D_{λ_1,λ_2} for $\lambda_2 \geq 0$:*

(1) If $\lambda_2 = 0$, then

$$\theta_{2,2}(D_{\lambda_1,0}) = \pi(1, (\lambda_1; 1), \Psi_3, 2, 2).$$

(2) If $\lambda_2 = 1$, then

$$\theta_{2,2}(D_{\lambda_1,1}) = \pi(1, (\lambda_1; 1), \Psi_3, 3, 1).$$

(3) If $\lambda_2 \geq 2$, then

$$\theta_{2,2}(D_{\lambda_1,\lambda_2}) = \pi((\lambda_1, \lambda_2; 2, 1), \Psi_6).$$

Notice that $\pi = \pi^\vee$ if π is an irreducible admissible representation of $Sp(p, q)$. On the other hand, $(P_{\lambda_1,\lambda_2})^\vee = P_{\lambda_1,\lambda_2}$, $(F_{\lambda_1,\lambda_2})^\vee = F_{\lambda_1,\lambda_2}$ and $(D_{\lambda_1,\lambda_2})^\vee = \overline{D}_{-\lambda_2,-\lambda_1}$. By the ten theorems and the occurrence sets given in this introduction, we determine theta (p, q) -lifts of all irreducible Harish-Chandra module of $O^*(4)$ explicitly, for all non-negative integers p and q .

2. Langlands Parameters and Lowest K -types

For a compact Lie group K , a K -type means an irreducible finite dimensional representation (up to isomorphism) of K . Let H be a real reductive group with a maximal compact subgroup K_H . We sometimes refer to K_H -types as K -types for H , or simply as K -types if the group H is clearly understood. We denote by $\mathcal{R}(H)$ the set of all isomorphism classes of irreducible Harish-Chandra modules of H . Since every Harish-Chandra module of H can be realized as the subspace of K_H -finite vectors of an admissible representation of H (see Corollary 4.2.4 of [26]), $\mathcal{R}(H)$ can also be regarded as the set of infinitesimal equivalence classes of irreducible admissible representations of H . By abuse of notation, for an isomorphism class (resp. an infinitesimal equivalence class) π in $\mathcal{R}(H)$, also let π denote an irreducible Harish-Chandra module (resp. an irreducible admissible representation) in this class. For every $\pi \in \mathcal{R}(H)$, we denote by $\mathcal{K}(\pi)$ the set of K_H -types occurring in π and by $\mathcal{A}(\pi)$ the set of lowest K_H -types of π (see Definition 5.1 of [22]). For $\pi \in \mathcal{R}(H)$ and $\sigma \in \mathcal{K}(\pi)$, $m(\sigma, \pi) = \dim \text{Hom}_{K_H}(\sigma, \pi)$ is a non-negative integer and we call it the multiplicity of σ in π .

Let \mathbb{H} be a quaternion algebra such that $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$ with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$ and $\mathbf{ki} = \mathbf{j}$. We define an involution on \mathbb{H} such that $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$.

In §2.1 and §2.2, we briefly review the Langlands parametrization for admissible dual of the groups $Sp(p, q)$ and $O^*(2n)$. We mainly follow the content in §2 of [11] and a more detailed discussion can be found in §3 of [15]. In §2.3, we introduce the algorithm for calculating the lowest K -types of all irreducible admissible representations of $Sp(p, q)$ and $O^*(2n)$. In §2.4, we state a more precise classification of admissible dual of the group $O^*(4)$. This work plays a key role in the explicit calculation of the local theta correspondence.

2.1. Parametrization for $\mathcal{R}(Sp(p, q))$

For every positive integer n , we denote by I_n the $n \times n$ identity matrix. For fixed non-negative integers p and q , let $G = Sp(p, q)$ be the isometry group of the Hermitian form $(,)$ on right \mathbb{H} column vector space \mathbb{H}^{p+q} given by

$$(v, w) = \sum_{i=1}^p \bar{v}_i w_i - \sum_{i=p+1}^{p+q} \bar{v}_i w_i \quad (1)$$

for $v, w \in \mathbb{H}^{p+q}$. Then

$$Sp(p, q) = \{g \in GL(p+q, \mathbb{H}) \mid g^* K_{p,q} g = K_{p,q}\}$$

with

$$K_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Here $g^* = \bar{g}^T$. The group $Sp(p, q)$ has a natural left action on the right \mathbb{H} column vector space \mathbb{H}^{p+q} . Let

$$K = \{(g_1, g_2) \in GL(p, \mathbb{H}) \times GL(q, \mathbb{H}) \mid g_1^* I_p g_1 = I_p, g_2^* I_q g_2 = I_q\}. \quad (2)$$

We know that $K = Sp(p) \times Sp(q)$ is a maximal compact subgroup of $Sp(p, q)$ (see §I.1 of [4]). We denote by S^1 the set of complex numbers whose absolute values are one. Let

$$T = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{p+q} \end{pmatrix} \mid a_i \in S^1 \right\}.$$

Then T is a maximal Cartan subgroup of K . The real Lie algebra \mathfrak{g}_0 of G is

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A + A^* = D + D^* = 0, C = B^* \right\} \quad (3)$$

with $A \in M_{p \times p}(\mathbb{H})$, $B \in M_{p \times q}(\mathbb{H})$, $C \in M_{q \times p}(\mathbb{H})$ and $D \in M_{q \times q}(\mathbb{H})$.

For every positive integer n , there is an injective homomorphism from $GL(n, \mathbb{H})$ (resp. $M_{n \times n}(\mathbb{H})$) into $GL(2n, \mathbb{C})$ (resp. $M_{2n \times 2n}(\mathbb{C})$):

$$A + B\mathbf{j} \mapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \quad (4)$$

with $A, B \in M_{n \times n}(\mathbb{C})$. By this homomorphism, the group $Sp(p, q)$ is embedded into $GL(2(p+q), \mathbb{C})$ as a closed subgroup and the real Lie algebra \mathfrak{g}_0 is isomorphic to

$$\left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mid A^* K_{p,q} + K_{p,q} A = 0, B^t K_{p,q} = K_{p,q} B \right\}$$

with $A, B \in M(p+q, \mathbb{C})$. Here

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

with $A_1 \in M_p(\mathbb{C})$, $A_4 \in M_q(\mathbb{C})$, $A_2 \in M_{p \times q}(\mathbb{C})$ and $A_3 \in M_{q \times p}(\mathbb{C})$. We know that

$$A_1 + A_1^* = 0, A_4 + A_4^* = 0, A_3 = A_2^*.$$

Similarly,

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

with $B_1 \in M_p(\mathbb{C})$, $B_4 \in M_q(\mathbb{C})$, $B_2 \in M_{p \times q}(\mathbb{C})$ and $B_3 \in M_{q \times p}(\mathbb{C})$. We know that

$$B_1 = B_1^t, B_4 = B_4^t, B_3 = -B_2^t.$$

The real Lie algebra \mathfrak{k}_0 of K is the set of matrices in \mathfrak{g}_0 such that $A_2 = B_2 = 0$ and the real Lie algebra \mathfrak{t}_0 of T is the set of diagonal matrices in \mathfrak{g}_0 . Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition of \mathfrak{g}_0 . We omit the subscript 0 to denote the complexified Lie algebras. We choose a basis $\{e_1, \dots, e_p; f_1, \dots, f_q\}$ of \mathfrak{it}_0^* such that the set of compact roots is $\Delta_c = \{\pm 2e_i, \pm 2f_i, \pm(e_i \pm e_j), \pm(f_i \pm f_j)\}$ and the set of non-compact roots is $\Delta_n = \{\pm(e_i \pm f_j)\}$. We choose a system of positive compact roots $\Delta_c^+ = \{2e_i, 2f_i, e_i \pm e_j, f_i \pm f_j | i < j\}$. One-half the sum of positive compact roots is

$$\rho_c = (p, p-1, \dots, 1; q, q-1, \dots, 1).$$

The discrete series of G is parameterized by Harish-Chandra parameters λ as follows:

$$(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q)$$

with $a_i, b_j \in \mathbb{Z}, a_1 > a_2 > \dots > a_p \geq 1, b_1 > b_2 > \dots > b_q \geq 1$ and $a_i \neq b_j$ for all i, j .

A limit of discrete series of G is parameterized by a pair (λ, Ψ) , where λ is the Harish-Chandra parameter and Ψ is a system of positive roots containing Δ_c^+ . The parameter $\lambda = (\lambda_1; \lambda_2)$ is of the form

$$(\overbrace{a_1, \dots, a_1}^{m_1}, \overbrace{a_2, \dots, a_2}^{m_2}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{a_1, \dots, a_1}^{n_1}, \dots, \overbrace{a_k, \dots, a_k}^{n_k}), \quad (5)$$

where $a_i \in \mathbb{Z}, a_1 > a_2 > \dots > a_k > 0$ and $|m_i - n_i| \leq 1$ for all i . Here Ψ is a system of positive roots satisfying the following condition (also see condition F-1 of [24]):

- (A) $\left\{ \begin{array}{l} \text{The system of compact positive roots } \Delta_c^+ \text{ is contained in } \Psi; \\ \text{The Harish-Chandra parameter } \lambda \text{ is dominant with respect to } \Psi; \\ \text{If } \langle \lambda, \alpha \rangle = 0 \text{ for a simple root } \alpha \text{ in } \Psi, \text{ then } \alpha \text{ is non-compact.} \end{array} \right.$

Consequently, there are 2^t different systems of positive roots satisfying condition (A), where t is the number of i such that $m_i = n_i > 0$ (c.f. §3.1 of [15]). We say λ satisfies condition (B) if $t = 0$. If λ satisfies condition (B), there is a unique system of positive roots Ψ satisfying condition (A). We denote by $\pi(\lambda, \Psi)$ the limit of discrete series parameterized by (λ, Ψ) . Its unique lowest K -type is $\Lambda = \lambda + \rho_n - \rho_c$, where ρ_n is one-half the sum of non-compact roots in Ψ (We identify K -types with their highest weight with respect to Δ_c^+).

Cuspidal parabolic subgroups of $Sp(p, q)$ are of the form $P = MAN = LN$ with

$$L \cong Sp(p-r, q-r) \times GL(1, \mathbb{H})^r \quad (6)$$

and $0 \leq r \leq \min\{p, q\}$. In fact, set $R = \min\{p, q\}$. Then $(\mathbb{H}^{p+q}, (\cdot, \cdot))$ admits a polar decomposition

$$\mathbb{H}^{p+q} = V_R^+ \oplus V^0 \oplus V_R^-, \quad (7)$$

where $V_R^+ = \text{span}_{\mathbb{H}}\{v_1, v_2, \dots, v_R\}$ and $V_R^- = \text{span}_{\mathbb{H}}\{v'_1, v'_2, \dots, v'_R\}$ are two maximal isotropic subspaces in \mathbb{H}^{p+q} and are dual to each other with respect to $(,)$. Here v_i, v'_i can be chosen such that $(v_i, v'_j) = \delta_{ij}$. For every integer r such that $0 \leq r \leq R$, we define

$$V_r^+ = \text{span}_{\mathbb{H}}\{v_1, v_2, \dots, v_r\}, \quad (8)$$

$$V_r^- = \text{span}_{\mathbb{H}}\{v'_1, v'_2, \dots, v'_r\}, \quad (9)$$

and define V^r to be

$$\text{span}_{\mathbb{H}}\{v_{R-r+1}, \dots, v_{R-1}, v_R\} \oplus V^0 \oplus \text{span}_{\mathbb{H}}\{v'_{R-r+1}, \dots, v'_{R-1}, v'_R\}. \quad (10)$$

Then

$$\{0\} \subset V_1^+ \subset V_2^+ \subset \dots \subset V_r^+ \quad (11)$$

is an isotropic flag in V_R^+ . The stabilizer P_r of this flag in $Sp(p, q)$ is a cuspidal parabolic subgroup of $Sp(p, q)$. Furthermore, we embed the group $Sp(p-r, q-r) \times GL(1, \mathbb{H})^r$ into P_r . The group $Sp(p-r, q-r)$ is regarded as the isometry group of V^{R-r} . The group $GL(1, \mathbb{H})^r$ is equipped with an left action \circ on \mathbb{H}^{p+q} which is right \mathbb{H} -linear as follows. For $h = (h_1, \dots, h_r) \in GL(1, \mathbb{H})^r$, the action of h on \mathbb{H}^{p+q} is

$$\begin{aligned} h \circ v_i &= v_i \cdot h_i, \quad \forall 1 \leq i \leq r, \\ h \circ v'_i &= v'_i \cdot (\overline{h_i})^{-1}, \quad \forall 1 \leq i \leq r, \\ h \circ u &= u, \quad \forall u \in V^{R-r}, \end{aligned} \quad (12)$$

where \cdot is the natural right multiplication on the right \mathbb{H} column vector space \mathbb{H}^{p+q} . It is easy to check that

$$(h \circ v_i, h \circ v'_i) = (v_i, v'_i)$$

for each $h \in GL(1, \mathbb{H})^r$ and each $1 \leq i \leq r$. Then we embed $Sp(p-r, q-r) \times GL(1, \mathbb{H})^r$ into the parabolic subgroup P_r . In general, let

$$\{0\} \subset V_{r_1}^+ \subset V_{r_2}^+ \subset \dots \subset V_{r_s}^+ \quad (13)$$

be an isotropic flag in V_R^+ , where $1 \leq r_1 < r_2 < \dots < r_s \leq R$. The stabilizer $P_{\{r_1, \dots, r_s\}}$ of this flag is a parabolic subgroup of $Sp(p, q)$. Set $r_0 = 0$ and $d_i = r_i - r_{i-1}$ for $1 \leq i \leq s$. The Levi subgroup of $P_{\{r_1, \dots, r_s\}}$ is isomorphic to

$$Sp(p-r_s, q-r_s) \times \prod_{i=1}^s GL(d_i, \mathbb{H}). \quad (14)$$

Up to conjugation, each proper parabolic subgroup of $Sp(p, q)$ is the stabilizer of an isotropic flag.

By the injective homomorphism (4), the group $GL(1, \mathbb{H})$ is

$$\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\} \cap GL(2, \mathbb{C}).$$

We define the determinant of an element in $GL(1, \mathbb{H})$ to be $|a|^2 + |b|^2$. For every non-negative integer m , we denote the $m+1$ -dimensional irreducible representation of $SU(2)$ by τ_m . Since

$GL(1, \mathbb{H}) \cong SU(2) \times \mathbb{R}$, every irreducible admissible representation of $GL(1, \mathbb{H})$ is of the form $\tau(\mu, \nu) = \tau_{\mu-1} \otimes \det(\cdot)^{\frac{\nu}{2}}$ with $\mu \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{C}$. The infinitesimal character of $\tau(\mu, \nu)$ is $(\frac{\nu+\mu}{2}, \frac{\nu-\mu}{2})$.

Given a cuspidal parabolic subgroups $P = LN$ of $Sp(p, q)$ with the Levi component as in (6), a limit of discrete series $\varpi = \pi(\lambda, \Psi)$ of $Sp(p-r, q-r)$ and a representation $\tau = \otimes_{i=1}^r \tau(\mu_i, \nu_i)$ of $GL(1, \mathbb{H})^r$, we call the normalized induced representation

$$Ind_p^{Sp(p,q)}(\varpi \otimes \tau \otimes \mathbb{1}) \quad (15)$$

a standard module and denote it by $X(P, r, \lambda, \Psi, \mu, \nu)$ with $\mu = (\mu_1, \dots, \mu_r) \in (\mathbb{Z}_{\geq 1})^r$ and $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{C}^r$. Since there is a unique cuspidal parabolic subgroup up to conjugation in $Sp(p, q)$ whose Levi component is isomorphic to $Sp(p-r, q-r) \times GL(1, \mathbb{H})^r$, for convenience, we usually omit the notation P and denote the standard module by $X(r, \lambda, \Psi, \mu, \nu)$. By abuse of notation, we also denote the underlying Harish-Chandra module of the standard module by $X(r, \lambda, \Psi, \mu, \nu)$. The non-parity condition F-2 in [24] amounts to the requirement that μ_i is odd if $\nu_i = 0$. If the non-parity condition is satisfied, the standard module $X(r, \lambda, \Psi, \mu, \nu)$ has a unique constituent containing all lowest $Sp(p) \times Sp(q)$ -types. We denote this constituent (infinitesimal equivalence class) by $\pi(r, \lambda, \Psi, \mu, \nu)$.

We identify the infinitesimal characters of Harish-Chandra modules (resp. admissible representations) of G with elements of the dual of a Cartan subalgebra of \mathfrak{g} (module the action of Weyl group), via the Harish-Chandra map. For $Sp(p, q)$, we choose the maximal compact Cartan subalgebra \mathfrak{t} for our Cartan subalgebra. Since the infinitesimal characters are preserved by normalized induction, the infinitesimal character of the standard module $X(r, \lambda, \Psi, \mu, \nu)$ and $\pi(r, \lambda, \Psi, \mu, \nu)$ is

$$(\lambda, \frac{\nu_1 + \mu_1}{2}, \frac{\nu_1 - \mu_1}{2}, \frac{\nu_2 + \mu_2}{2}, \frac{\nu_2 - \mu_2}{2}, \dots, \frac{\nu_r + \mu_r}{2}, \frac{\nu_r - \mu_r}{2}) \in \mathfrak{t}^* / \sim, \quad (16)$$

where \sim means the equivalence under the action of Weyl group W of $\Delta(\mathfrak{g} : \mathfrak{t})$.

For the classification of admissible dual of $Sp(p, q)$, we have the following theorem (see §2.2 of [11]).

Theorem 2.1. *Every irreducible admissible representation of $Sp(p, q)$ is infinitesimal equivalent to some $\pi(r, \lambda, \Psi, \mu, \nu)$ as mentioned above. Furthermore, $\pi(r, \lambda, \Psi, \mu, \nu)$ is infinitesimal equivalent to $\pi(r', \lambda', \Psi', \mu', \nu')$ $\Leftrightarrow r = r', \lambda = \lambda', \Psi = \Psi'$ and (μ', ν') may be obtained from (μ, ν) by simultaneous permutation of the coordinates, and by replacing some of the ν_i by $-\nu_i$.*

Thus we obtain a parametrization for $\mathcal{R}(Sp(p, q))$.

2.2. Parametrization for $\mathcal{R}(O^*(2n))$

For a fixed positive integer n , let $G = O^*(2n)$ be the isometry group of the skew-Hermitian form \langle, \rangle on left \mathbb{H} row vector space \mathbb{H}^n given by

$$\langle v, w \rangle = \sum_{i=1}^n v_i \mathbf{i} \bar{w}_i \quad (17)$$

for $v, w \in \mathbb{H}^n$. The group $GL(n, \mathbb{H})$ has a natural right action on \mathbb{H}^n . Then

$$O^*(2n) = \{g \in GL(n, \mathbb{H}) | g \mathbf{i} U_n g^* = \mathbf{i} U_n\}. \quad (18)$$

By (18), it is not difficult to see that $K = U(n)$ is naturally embedded into $O^*(2n)$ as a maximal compact group (see §I.1 of [5]). Furthermore, $T = U(1)^n$ is naturally embedded into $O^*(2n)$ and is regarded as a maximal Cartan subgroup of K . By the homomorphism (4), $O^*(2n)$ can be embedded into $GL(2n, \mathbb{C})$ as a closed subgroup, while

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \mid A \in U(n) \right\}. \quad (19)$$

The real Lie algebra $\mathfrak{g}_0 = \mathfrak{o}^*(2n)$ of $O^*(2n)$ is

$$\{g \in M_{n \times n}(\mathbb{H}) \mid g\mathbf{i}I_n + \mathbf{i}I_n g^* = 0\}.$$

By the homomorphism (4), the real Lie algebra \mathfrak{g}_0 is isomorphic to

$$\left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \mid A + A^* = 0, B + B^t = 0 \right\} \quad (20)$$

with $A, B \in M_{n \times n}(\mathbb{C})$. The real Lie algebra \mathfrak{k}_0 of K is the set of matrices in \mathfrak{g}_0 such that $B = 0$ and the real Lie algebra \mathfrak{t}_0 of T is the set of diagonal matrices in \mathfrak{g}_0 . Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition. We omit the subscript 0 to denote the complexified Lie algebras.

We choose a basis $\{e_1, \dots, e_n\}$ of $i\mathfrak{t}_0^*$ such that the set of compact roots is $\Delta_c = \{\pm(e_i - e_j) \mid i < j\}$ and the set of non-compact roots is $\Delta_n = \{\pm(e_i + e_j) \mid i < j\}$. We choose a system of positive compact roots $\Delta_c^+ = \{e_i - e_j \mid i < j\}$. One-half the sum of positive compact roots is

$$\rho_c = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2} \right).$$

The discrete series of G is parameterized by Harish-Chandra parameters λ as follows:

$$(a_1, a_2, \dots, a_n)$$

with $a_i \in \mathbb{Z}, a_1 > a_2 > \dots > a_n$ and $a_i + a_j \neq 0$.

A limit of discrete series of G is parameterized by a pair (λ, Ψ) , where λ is the Harish-Chandra parameter and Ψ is a positive root system containing Δ_c^+ . The parameter λ is of the form

$$\left(\overbrace{a_1, \dots, a_1}^{m_1}, \overbrace{a_2, \dots, a_2}^{m_2}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1} \right) \quad (21)$$

with a_i as in (5); or with one zero between the last a_k and the first $-a_k$. Here Ψ is a system of positive roots satisfying condition (A) as for $Sp(p, q)$. Consequently, there are 2^t different systems of positive roots satisfying condition (A), where t is the number of i such that $0 < m_i = n_i$. If λ satisfies condition (B) as in §2.1, there is a unique system of positive roots Ψ satisfying condition (A). As in §2.1, we denote by $\pi(\lambda, \Psi)$ the limit of discrete series parameterized by (λ, Ψ) . Its unique lowest K -type is $\Lambda = \lambda + \rho_n - \rho_c$, where ρ_n is one-half the sum of non-compact roots in Ψ .

Cuspidal parabolic subgroups of $O^*(2n)$ are of the form $P = MAN = LN$ with

$$L \cong O^*(2(n-2r)) \times GL(1, \mathbb{H})^r \quad (22)$$

and $0 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Set $R = \lfloor \frac{n}{2} \rfloor$. Then $(\mathbb{H}^n, <, >)$ admits a polar decomposition

$$\mathbb{H}^n = V_R^+ \oplus V^0 \oplus V_R^-, \quad (23)$$

where $V_R^+ = \text{span}_{\mathbb{H}}\{v_1, v_2, \dots, v_R\}$ and $V_R^- = \text{span}_{\mathbb{H}}\{v'_1, v'_2, \dots, v'_R\}$ are two maximal isotropic subspaces in \mathbb{H}^n and are dual to each other with respect to \langle, \rangle . Here v_i, v'_i can be chosen such that $\langle v_i, v'_j \rangle = \delta_{ij}$. For every integer r such that $0 \leq r \leq R$, we define V_r, W_r and X_r as in (8), (9) and (10). Then

$$\{0\} \subset V_1^+ \subset V_2^+ \subset \dots \subset V_r^+ \quad (24)$$

is an isotropic flag in V_R^+ . The stabilizer P_r of this flag in $O^*(2n)$ is a cuspidal parabolic subgroup of $O^*(2n)$. Furthermore, the group $O^*(2(n-2r)) \times GL(1, \mathbb{H})^r$ can be embedded into P_r . The group $O^*(2(n-2r))$ is regarded as the isometry group of V^{R-r} . The group $GL(1, \mathbb{H})^r$ is equipped with an right action \circ on \mathbb{H}^n which is left \mathbb{H} -linear as follows. For $h = (h_1, \dots, h_r) \in GL(1, \mathbb{H})^r$, the action of h on \mathbb{H}^n is

$$\begin{aligned} v_i \circ h &= (\overline{h_i})^{-1} \cdot v_i, \quad \forall 1 \leq i \leq r, \\ v'_i \circ h &= h_i \cdot v'_i, \quad \forall 1 \leq i \leq r, \\ u \circ h &= u, \quad \forall u \in V^{R-r}, \end{aligned} \quad (25)$$

where \cdot is the natural left multiplication on the left \mathbb{H} row vector space \mathbb{H}^n . It is easy to check that

$$(v_i \circ h, v'_i \circ h) = (v_i, v'_i)$$

for each $h \in GL(1, \mathbb{H})^r$ and each $1 \leq i \leq r$. Then we embed $O^*(2(n-2r)) \times GL(1, \mathbb{H})^r$ into the parabolic subgroup P_r . In general, let

$$\{0\} \subset V_{r_1}^+ \subset V_{r_2}^+ \subset \dots \subset V_{r_s}^+ \quad (26)$$

be an isotropic flag in V , where $1 \leq r_1 < r_2 < \dots < r_s \leq R$. The stabilizer $P_{\{r_1, \dots, r_s\}}$ of this flag is a parabolic subgroup of $O^*(2n)$. Set $r_0 = 0$ and $d_i = r_i - r_{i-1}$ for $1 \leq i \leq s$. Its Levi subgroup is isomorphic to

$$O^*(2(n-2r_s)) \times \prod_{i=1}^s GL(d_i, \mathbb{H}). \quad (27)$$

Here the group $GL(d_i, \mathbb{H})$ has a right action on the left row \mathbb{H} space

$$\text{span}_{\mathbb{H}}\{v'_{r_{i-1}+1}, v'_{r_{i-1}+2}, \dots, v'_{r_i}\}$$

as the natural right multiplication and a right action on the left \mathbb{H} space

$$\text{span}_{\mathbb{H}}\{v_{r_{i-1}+1}, v_{r_{i-1}+2}, \dots, v_{r_i}\}$$

such that $\langle v_k \circ h, v'_l \circ h \rangle = \langle v_k, v'_l \rangle$ for $r_{i-1} < k, l \leq r_i$ and $h \in GL(d_i, \mathbb{H})$. Up to conjugation, each proper parabolic subgroup of $O^*(2n)$ is the stabilizer of an isotropic flag.

Substituting $Sp(p-r, q-r)$ by $O^*(2(n-2r))$, we define $X(r, \lambda, \Psi, \mu, \nu)$, $\pi(r, \lambda, \Psi, \mu, \nu)$ and obtain a parametrization for $\mathcal{R}(O^*(2n))$ as in §2.1. For $O^*(2n)$, the non-parity condition F-2 also amounts to the requirement that μ_i is odd if $\nu_i = 0$. The infinitesimal character of $X(r, \lambda, \Psi, \mu, \nu)$ and $\pi(r, \lambda, \Psi, \mu, \nu)$ is similar to that for $Sp(p, q)$.

2.3. The Lowest K -types

Using the standard theory of [5] and [23], we compute the lowest K -types of irreducible admissible representations of the groups $Sp(p, q)$ and $O^*(2n)$ from the Langlands parameters. The algorithm can also be founded in §2.3 of [11] and §3 of [15].

Let $G = Sp(p, q)$ and let $K = Sp(p) \times Sp(q)$ be the maximal compact subgroup of G as in (2). Let T and Δ_c^+ be those in §2.1. We identify a K -type σ with its highest weight with respect to Δ_c^+ , also denoted by σ . By definition 5.1 of [22], let $\|\cdot\|_K$ be the norm of K -types as follows: for a K -type σ , $\|\sigma\|_K = \langle \sigma + 2\rho_c, \sigma + 2\rho_c \rangle$, where \langle, \rangle is the Killing form and ρ_c is that in §2.1. A K -types σ is of the form

$$(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q)$$

with $a_1 \geq a_2 \geq \dots \geq a_p \geq 0$ and $b_1 \geq b_2 \geq \dots \geq b_q \geq 0$. The norm of σ is

$$\|\sigma\|_K = \sum_{i=1}^p (a_i + 2p + 2 - 2i)^2 + \sum_{j=1}^q (b_j + 2q + 2 - 2j)^2.$$

If a K -type σ occurs with minimal norm in an admissible representation π of G , we call σ a lowest K -type of π .

Let $\pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $Sp(p, q)$, with

$$\lambda = (\lambda_1; \lambda_2) = (\overbrace{a_1, \dots, a_1}^{m_1}, \overbrace{a_2, \dots, a_2}^{m_2}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}; \overbrace{a_1, \dots, a_1}^{n_1}, \dots, \overbrace{a_k, \dots, a_k}^{n_k})$$

as in (5). For $\pi(r, \lambda, \Psi, \mu, \nu)$, we write λ' as the dominant weight which is W -conjugate to $(\lambda_1, \frac{\mu}{2}; \lambda_2, \frac{\mu}{2})$. Write

$$\lambda' = (\overbrace{\alpha_1, \dots, \alpha_1}^{M_1}, \overbrace{\alpha_2, \dots, \alpha_2}^{M_2}, \dots, \overbrace{\alpha_s, \dots, \alpha_s}^{M_s}; \overbrace{\alpha_1, \dots, \alpha_1}^{N_1}, \dots, \overbrace{\alpha_s, \dots, \alpha_s}^{N_s}) \quad (28)$$

with $\alpha_1 > \alpha_2 > \dots > \alpha_s > 0$. Notice that $\alpha_i \in \frac{1}{2}\mathbb{Z}$ and $|M_i - N_i| \leq 1$. If $M_i \neq N_i$, α_i is an integer.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ -stable parabolic subalgebra of \mathfrak{g} defined by λ' (see Definition 5.2.1 of [23]). The lowest K -types of $\pi(r, \lambda, \Psi, \mu, \nu)$ are of the form

$$\Lambda = \lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L. \quad (29)$$

Here $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are one-half of the sums of roots associated to $\mathfrak{u} \cap \mathfrak{p}$ and $\mathfrak{u} \cap \mathfrak{k}$, respectively, and δ_L will be given explicitly below.

Set $\tilde{M}_i = M_1 + \dots + M_i$ and $\tilde{N}_i = N_1 + \dots + N_i$. Then $\tilde{M}_s = p$, $\tilde{N}_s = q$ and we write $\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ from (29) as

$$(\overbrace{\beta_1, \dots, \beta_1}^{M_1}, \dots, \overbrace{\beta_s, \dots, \beta_s}^{M_s}; \overbrace{\gamma_1, \dots, \gamma_1}^{N_1}, \dots, \overbrace{\gamma_s, \dots, \gamma_s}^{N_s}), \quad (30)$$

where

$$\begin{aligned} \beta_i &= \alpha_i + \tilde{M}_i - \tilde{N}_i - \frac{1}{2}(M_i - N_i + 1) + q - p, \\ \gamma_i &= \alpha_i - \tilde{M}_i + \tilde{N}_i - \frac{1}{2}(N_i - M_i + 1) + p - q. \end{aligned} \quad (31)$$

Theorem 2.2. *Let $\pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $Sp(p, q)$. Write λ as in (5) with k , a_i , m_i and n_i as defined there. Write $\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ from (29) as in (30) and (31). Let $\tilde{m}_j = \sum_{c=1}^j m_c$ and let $\tilde{n}_j = \sum_{c=1}^j n_c$ for $1 \leq j \leq k$. Let $\tilde{m}_0 = \tilde{n}_0 = 0$. The lowest K -types of $\pi(r, \lambda, \Psi, \mu, \nu)$ are precisely those of the form*

$$\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L$$

with

$$\delta_L = (\overbrace{\delta_1, \dots, \delta_1}^{M_1}, \dots, \overbrace{\delta_s, \dots, \delta_s}^{M_s}, \overbrace{-\delta_1, \dots, -\delta_1}^{N_1}, \dots, \overbrace{-\delta_s, \dots, -\delta_s}^{N_s})$$

satisfying the following condition:

- (1) If β_i is an integer, then $\delta_i = 0$.
- (2) Suppose $\beta_i \in \mathbb{Z} + \frac{1}{2}$, then $\delta_i = \frac{1}{2}$ or $-\frac{1}{2}$. If α_i does not occur as an entry of in λ , then both choices occur. If $\alpha_i = a_j$, then $\delta_i = \frac{1}{2}$ if $e_{\tilde{m}_j-1+1} - f_{\tilde{n}_j-1+1} \in \Psi$, and $\delta_i = -\frac{1}{2}$ otherwise.

Remark 2.3. By the definition of $\pi(r, \lambda, \Psi, \mu, \nu)$, its lowest $Sp(p) \times Sp(q)$ -types coincide with those of the standard module $X(r, \lambda, \Psi, \mu, \nu)$. In fact, every lowest $Sp(p) \times Sp(q)$ -type of the standard module occurs with multiplicity one (see §3.2 of [14]).

Let $G = O^*(2n)$ and let $K = U(n)$ be the maximal compact subgroup as in (19). Let T, Δ_c^+ and ρ_c be those in §2.2. Let \mathfrak{t}_0 be the real Lie algebra of T as in §2.2. For $O^*(2n)$, a K -type σ is of the form

$$(a_1, a_2, \dots, a_n)$$

with $a_1 \geq a_2 \geq \dots \geq a_n$. The norm of σ is

$$\|\sigma\|_K = \langle \sigma + 2\rho_c, \sigma + 2\rho_c \rangle = \sum_{i=1}^n (a_i + n + 1 - 2i)^2.$$

We define the lowest K -types for admissible representations of $O^*(2n)$ as for those of $Sp(p, q)$.

Let $\pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $O^*(2n)$, with

$$\lambda = (\overbrace{a_1, \dots, a_1}^{m_1}, \overbrace{a_2, \dots, a_2}^{m_2}, \dots, \overbrace{a_k, \dots, a_k}^{m_k}, \overbrace{-a_k, \dots, -a_k}^{n_k}, \dots, \overbrace{-a_1, \dots, -a_1}^{n_1});$$

as in (21); or with one zero between the last a_k and the first $-a_k$. We write λ' as the dominant weight which is W -conjugate to $(\lambda, \frac{\mu}{2}, -\frac{\mu}{2})$. Write

$$\lambda' = (\overbrace{\alpha_1, \dots, \alpha_1}^{M_1}, \overbrace{\alpha_2, \dots, \alpha_2}^{M_2}, \dots, \overbrace{\alpha_s, \dots, \alpha_s}^{M_s}, \overbrace{-\alpha_s, \dots, -\alpha_s}^{N_s}, \dots, \overbrace{-\alpha_1, \dots, -\alpha_1}^{N_1}) \quad (32)$$

with α_i as in (28); or with one zero between the last α_s and the first $-\alpha_s$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ be the θ -stable parabolic subalgebra of \mathfrak{g} defined by λ' . The lowest K -types of $\pi(r, \lambda, \Psi, \mu, \nu)$ are of the form

$$\Lambda = \lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L. \quad (33)$$

Here $\rho(\mathfrak{u} \cap \mathfrak{p})$ and $\rho(\mathfrak{u} \cap \mathfrak{k})$ are one-half of the sums of roots associated to $\mathfrak{u} \cap \mathfrak{p}$ and $\mathfrak{u} \cap \mathfrak{k}$, respectively, and δ_L will be given explicitly below.

Set $\tilde{M}_i = M_1 + \dots + M_i$ and $\tilde{N}_i = N_1 + \dots + N_i$. Then $\tilde{M}_s + \tilde{N}_s = n$ or $n - 1$ and we write $\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ from (33) as

$$\gamma' = (\overbrace{\beta_1, \dots, \beta_1}^{M_1}, \dots, \overbrace{\beta_s, \dots, \beta_s}^{M_s}, \overbrace{-\gamma_s, \dots, -\gamma_s}^{N_s}, \dots, \overbrace{-\gamma_1, \dots, -\gamma_1}^{N_1}) \quad (34)$$

if $\tilde{M}_s + \tilde{N}_s = n$; or with $\tilde{M}_s - \tilde{N}_s$ between the last β_s and the first $-\gamma_s$ if $\tilde{M}_s + \tilde{N}_s = n - 1$. Here

$$\begin{aligned} \beta_i &= \alpha_i + \tilde{M}_i - \tilde{N}_i - \frac{1}{2}(M_i - N_i + 1), \\ \gamma_i &= \alpha_i - \tilde{M}_i + \tilde{N}_i - \frac{1}{2}(N_i - M_i + 1). \end{aligned} \quad (35)$$

Theorem 2.4. Let $\pi(r, \lambda, \Psi, \mu, \nu)$ be an irreducible admissible representation of $O^*(2n)$. Write λ as in (21) with k, a_i, m_i and n_i as defined there. Write $\lambda' + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k})$ from (33) as in (34) and (35). Let $\tilde{m}_j = \sum_{c=1}^j m_j$ and let $\tilde{n}_j = n - 2r - \sum_{c=1}^j n_j$ for $1 \leq j \leq k$. Let $\tilde{m}_0 = 0$ and let $\tilde{n}_0 = n - 2r$. The lowest K -types of $\pi(r, \lambda, \Psi, \mu, \nu)$ are precisely those of the form

$$\Lambda = \gamma + \rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u} \cap \mathfrak{k}) + \delta_L.$$

with

$$\delta_L = (\overbrace{\delta_1, \dots, \delta_1}^{M_1}, \dots, \overbrace{\delta_s, \dots, \delta_s}^{M_s}, \overbrace{\delta_s, \dots, \delta_s}^{N_s}, \dots, \overbrace{\delta_1, \dots, \delta_1}^{N_1})$$

if $\tilde{M}_s + \tilde{N}_s = n$; or with one zero as the $(\tilde{M}_s + 1)$ -th entry if $\tilde{M}_s + \tilde{N}_s = n - 1$. Here δ_L is any element in \mathfrak{it}_0^* satisfying the following condition:

- (1) If β_i is an integer, then $\delta_i = 0$.
- (2) Suppose $\beta_i \in \mathbb{Z} + \frac{1}{2}$, then $\delta_i = \frac{1}{2}$ or $-\frac{1}{2}$. If α_i does not occur as an entry of in λ , then both choices occur. If $\alpha_i = a_j$, then $\delta_i = \frac{1}{2}$ if $e_{\tilde{m}_{j-1}+1} + e_{\tilde{n}_j+1} \in \Psi$, and $\delta_i = -\frac{1}{2}$ otherwise.

For $O^*(2n)$, we have the similar conclusions as in Remark 2.3.

2.4. Admissible Dual of $O^*(4)$

The classification of the admissible dual of the groups $O^*(2n)$ has been given in §2.2. In this section, we will give a more accurate description of the admissible dual of $O^*(4)$. As in (20), the real Lie algebra $\mathfrak{o}^*(4)$ is

$$\left\{ \begin{pmatrix} A & B \\ -\overline{B} & A \end{pmatrix} \mid A + A^* = 0, B + B^t = 0 \right\}$$

with $A, B \in M_{2 \times 2}(\mathbb{C})$. Let φ_1 be an injective homomorphism from $\mathfrak{su}(2)$ into $\mathfrak{o}^*(4)$:

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.$$

Let φ_2 be an injective homomorphism from $\mathfrak{sl}(2, \mathbb{R})$ into $\mathfrak{o}^*(4)$:

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &\mapsto \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}, Y = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}. \end{aligned}$$

Then $\mathfrak{o}^*(4)$ is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{R})$. As in (19), $U(2)$ can be regarded as a maximal compact subgroup of $O^*(4)$. Then the homomorphism φ_1 can be lifted to an embedding Φ_1 of $SU(2)$ into $O^*(4)$ and the homomorphism φ_2 can be lifted to an embedding Φ_2 of $SL(2, \mathbb{R})$ into $O^*(4)$ (The homomorphism φ_2 restricted on the maximal compact subalgebra $\mathfrak{so}(2)$ of $\mathfrak{sl}(2, \mathbb{R})$ can be lifted to an isomorphism from $SO(2)$ to $U(1)$). So we have a projection map $\Phi = \Phi_1 \times \Phi_2$ from $SU(2) \times SL(2, \mathbb{R})$ onto $O^*(4)$. As in §5 of [12], we identify $G = O^*(4)$ with $(SU(2) \times SL(2, \mathbb{R}))/\{\pm I\}$. To understand the representations of $O^*(4)$, it is necessary for us to understand the representations of $SU(2)$ and $SL(2, \mathbb{R})$.

Let $G_1 = SU(2)$ and let $G_2 = SL(2, \mathbb{R})$. All irreducible admissible representations of $SU(2)$ are of the form τ_k , where τ_k is the $k + 1$ -dimensional irreducible representation of $SU(2)$. Then we concentrate on the admissible dual of $SL(2, \mathbb{R})$. The classification can be founded in [4] and [7] and we state the results in this section.

The unique proper parabolic subgroup of G_2 up to conjugation is the upper triangular group $P_2 = M_2 A_2 N_2$, where $M_2 = \{\pm I\}$,

$$A_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a > 0 \right\},$$

and

$$N_2 = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{R} \right\}.$$

For $k \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$, we denote by $I^k(\nu)$ the induced representation $Ind_{P_2}^{SL(2, \mathbb{R})}(\text{sgn}^k \otimes \nu \otimes \mathbb{1})$. For $\nu \in \mathbb{Z}_{\geq 1}$, we denote by D_ν the irreducible lowest weight $SL(2, \mathbb{R})$ -module of lowest weight ν , and by $\overline{D}_{-\nu}$ the irreducible highest weight $SL(2, \mathbb{R})$ -module of highest weight $-\nu$. For $\nu \in \mathbb{Z}_{\geq 0}$, we denote by F_ν the irreducible finite dimensional representation of $SL(2, \mathbb{R})$ of highest weight ν (or dimension $\nu + 1$). We have the following theorems (see Chapter VI of [7]).

Theorem 2.5. (1) If $k \not\equiv \nu - 1 \pmod{2}$, then $I^k(\nu)$ is an irreducible representation. Furthermore, $I^k(\nu)$ is infinitesimal equivalent to $I^k(-\nu)$.
(2) If $k \equiv \nu - 1 \pmod{2}$ and $\nu > 0$, then $I^k(\nu)$ has two irreducible submodule: $D_{\nu+1}$ and $\overline{D}_{-(\nu+1)}$. The unique irreducible quotient of $I^k(\nu)$ is $F_{\nu-1}$.
(3) If $k \equiv \nu - 1 \pmod{2}$ and $\nu > 0$, then $I^k(-\nu)$ has a unique submodule $F_{\nu-1}$, and its quotient by $F_{\nu-1}$ is isomorphic to $D_{\nu+1} \oplus \overline{D}_{-(\nu+1)}$.
(4) If k is odd, then $I^k(0)$ is isomorphic to $D_1 \oplus \overline{D}_{-1}$.

Theorem 2.6. The following is a list of infinitesimal equivalence classes of irreducible admissible representations of $SL(2, \mathbb{R})$:

- (1) Irreducible principal series $I^k(\nu)$, $\text{Re } \nu \geq 0$;
- (2) Lowest weight modules $D_{\nu+1}$, $\nu \in \mathbb{Z}_{\geq 0}$;
- (3) Highest weight modules $\overline{D}_{-(\nu+1)}$, $\nu \in \mathbb{Z}_{\geq 0}$;
- (4) Finite dimensional representations $F_{\nu-1}$, $\nu \in \mathbb{Z}_{\geq 1}$.

Let $P = (G_1 \times P_2)/\{\pm I\}$. It is not difficult to check that P is the stabilizer of the isotropic row vector space $\mathbb{H}(1, -\mathbf{j})$ in $O^*(4)$ as in §2.2, where $\mathbb{H}^2 = \mathbb{H}(1, -\mathbf{j}) \oplus \mathbb{H}(\frac{1}{2}, \frac{\mathbf{k}}{2})$ such that $\langle (1, -\mathbf{j}), (\frac{1}{2}, \frac{\mathbf{k}}{2}) \rangle = 1$. For $k \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$, it is not difficult to see that the representation $\tau_k \otimes I^k(\nu)$ is infinitesimal equivalent to the standard module $X(1, 0, \emptyset, k + 1, \nu)$ as a representation of $SU(2) \times SL(2, \mathbb{R})$ (also see §5 of [12]). Furthermore, if ν is a non-negative integer such that $\nu \equiv k + 1 \pmod{2}$, $\tau_k \otimes D_{\nu+1}$ is infinitesimal equivalent to the limit of discrete series $\pi((\frac{\nu+k+1}{2}, \frac{\nu-k-1}{2}), \{e_1 \pm e_2\})$ while $\tau_k \otimes \overline{D}_{-(\nu+1)}$ is infinitesimal equivalent to the limit of discrete series $\pi((\frac{-\nu+k+1}{2}, \frac{-\nu-k-1}{2}), \{e_2 \pm e_1\})$ (as representations of $SU(2) \times SL(2, \mathbb{R})$). If $\nu \in \mathbb{Z}_{\geq 1}$, by Theorem 2.5, it is not difficult to see that the unique irreducible quotient of $X(1, 0, \emptyset, k + 1, \nu)$ is infinitesimal equivalent to $\tau_k \otimes F_{\nu-1}$ as a representation of $SU(2) \times SL(2, \mathbb{R})$, and this quotient is infinitesimal equivalent to $\pi(1, 0, \emptyset, k + 1, \nu)$ as a representation of $O^*(4)$. The following is a list of infinitesimal equivalence classes of irreducible admissible representations of $O^*(4)$:

- (1) Irreducible principal series $\pi(1, 0, \emptyset, k + 1, \nu)$, $\text{Re } \nu \geq 0$, $\nu \not\equiv k + 1 \pmod{2}$;

- (2) Limits of lowest weight discrete series $\pi((\frac{\nu+k+1}{2}, \frac{\nu-k-1}{2}), \Psi)$ with $\Psi = \{e_1 - e_2, e_1 + e_2\}$, $\nu \in \mathbb{Z}_{\geq 0}$ and $\nu \equiv k + 1 \pmod{2}$;
- (3) Limits of highest weight discrete series $\pi((\frac{-\nu+k+1}{2}, \frac{-\nu-k-1}{2}), \Psi)$ with $\Psi = \{e_1 - e_2, -(e_1 + e_2)\}$, $\nu \in \mathbb{Z}_{\geq 0}$ and $\nu \equiv k + 1 \pmod{2}$;
- (4) Finite dimensional representations $\pi(1, 0, \emptyset, k + 1, \nu)$, $\nu \in \mathbb{Z}_{\geq 1}$, $\nu \equiv k + 1 \pmod{2}$.

It is easy to check the infinitesimal equivalence classes in each case are uniquely determined by their infinitesimal characters $(\lambda_1, \lambda_2) = (\frac{\nu+k+1}{2}, \frac{\nu-k-1}{2})$. By results in Chapter VI of [7], it is not difficult to determine their structures of $U(2)$ -types. We have the following theorem.

Theorem 2.7. *The following is a list of infinitesimal equivalence classes of irreducible admissible representations of $O^*(4)$:*

- (1) Irreducible principal series $P_{\lambda_1, \lambda_2} = \pi(1, 0, \emptyset, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$ with $\lambda_1, \lambda_2 \notin \mathbb{Z}$, $\text{Re}(\lambda_1 + \lambda_2) \geq 0$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$. Here $\mathcal{K}(P_{\lambda_1, \lambda_2}) = \{(p + \lambda_1 - \lambda_2 - 1, p) | \forall p \in \mathbb{Z}\}$;
- (2) Limits of lowest weight discrete series $D_{\lambda_1, \lambda_2} = \pi((\lambda_1, \lambda_2), \Psi)$ with $\Psi = \{e_1 - e_2, e_1 + e_2\}$. Here $\mathcal{K}(D_{\lambda_1, \lambda_2}) = \{(p + \lambda_1 - \lambda_2 - 1, p) | \forall p \geq \lambda_2 + 1\}$;
- (3) Limits of highest weight discrete series $\overline{D}_{\lambda_1, \lambda_2} = \pi((\lambda_1, \lambda_2), \Psi)$ with $\Psi = \{e_1 - e_2, -(e_1 + e_2)\}$. Here $\mathcal{K}(\overline{D}_{\lambda_1, \lambda_2}) = \{(p + \lambda_1 - \lambda_2 - 1, p) | \forall p \leq \lambda_2\}$;
- (4) Finite dimensional representations $F_{\lambda_1, \lambda_2} = \pi(1, 0, \emptyset, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\geq 1}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$. Here $\mathcal{K}(F_{\lambda_1, \lambda_2}) = \{(p + \lambda_1 - \lambda_2 - 1, p) | 1 - \lambda_1 \leq p \leq \lambda_2\}$.

Furthermore, for each $\pi \in \mathcal{R}(O^*(4))$, the multiplicity $m(\sigma, \pi)$ of each $\sigma \in \mathcal{K}(\pi)$ equals to one.

3. Local Theta Correspondence

For non-negative integers p, q, n such that $p + q \geq 1$ and $n \geq 1$, let $V = \mathbb{H}^{p+q}$ be a right \mathbb{H} column vector space with Hermitian form (1) and let $V' = \mathbb{H}^n$ be a left \mathbb{H} row vector space with skew-Hermitian form (17). Let $W = \mathbb{H}^{p+q} \otimes_{\mathbb{H}} \mathbb{H}^n$ be a $4n(p + q)$ -dimensional real space with symplectic form $\langle \cdot, \cdot \rangle = \text{tr}_{\mathbb{H}/\mathbb{R}}((\cdot, \cdot) \otimes \overline{\langle \cdot, \cdot \rangle})$. As in §2, the group $Sp(p, q)$ has a left action on \mathbb{H}^{p+q} and the group $O^*(2n)$ has a right action on \mathbb{H}^n . Then the group $Sp(p, q) \times O^*(2n)$ has a left action on W as follows. For $g \in Sp(p, q)$, $h \in O^*(2n)$, $v \in \mathbb{H}^{p+q}$ and $w \in \mathbb{H}^n$,

$$(g, h) \cdot (v \otimes w) = (g \cdot v) \otimes (w \cdot h^{-1}). \quad (36)$$

Since this action keeps the symplectic form $\langle \cdot, \cdot \rangle$ invariant, $(Sp(p, q), O^*(2n))$ is embedded into $Sp(4n(p + q), \mathbb{R})$. Furthermore, $Sp(p, q)$ and $O^*(2n)$ are centralizer of each other (as algebraic groups) in the symplectic group $Sp = Sp(4n(p + q), \mathbb{R})$. We call $(Sp(p, q), O^*(2n))$ a reductive dual pair in Sp . We endow W with a complex Hermitian form as follows. Let V_1 be the subspace of vectors of V whose last q entries are zero and let V_2 be the subspace of vectors of V whose first p entries are zero. We define a complex multiplication $*$ on W as follows:

$$\begin{aligned} c * (v_1 \otimes v') &= v_1 \otimes v' \cdot c, \quad \forall c \in \mathbb{C}, \quad \forall v_1 \in V_1, \quad \forall v' \in V', \\ c * (v_2 \otimes v') &= v_2 \otimes v' \cdot \overline{c}, \quad \forall c \in \mathbb{C}, \quad \forall v_2 \in V_2, \quad \forall v' \in V'. \end{aligned} \quad (37)$$

Here \cdot is the natural right multiplication on \mathbb{H}^n . We define a norm $\|\cdot\|$ on W : $\|w\|^2 = \langle w, \mathbf{i} * w \rangle$. The isometry group U of this norm is a maximal compact subgroup of Sp and is isomorphic to $U(2n(p + q))$. By the action (36), the maximal compact group $K = Sp(p) \times Sp(q)$ of $G = Sp(p, q)$

as in (2) and the maximal compact subgroup $K' = U(n)$ of $G' = O^*(2n)$ as in (19) act on W . It is easy to check that the actions of K and K' on W commute with the complex multiplication and keep the norm invariant. Then K and K' can be regarded as compact subgroups of U .

For a fixed additive character $\psi(t) = e^{2\pi i t}$ of \mathbb{R} and the two-fold metaplectic cover group \widetilde{Sp} of Sp , a unitary representation of \widetilde{Sp} associated to ψ was constructed by Shale [19] and Weil [27]. We call this representation the oscillator representation. We denote by $\omega_{p,q,n}$ the oscillator representation for the reductive dual pair $(Sp(p, q), O^*(2n))$. Let \widetilde{U} denote the inverse image of U in the metaplectic cover and let $\mathcal{F}_{p,q,n}$ denote the Fock space (the Harish-Chandra module) of $\omega_{p,q,n}$. Since the two-fold metaplectic groups of both $Sp(p, q)$ and $O^*(2n)$ uniquely split, we can state the results in terms of the groups themselves instead of their covering groups. We denote by $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$ (resp. $\mathcal{R}(O^*(2n), \omega_{p,q,n})$) the set of elements in $\mathcal{R}(Sp(p, q))$ (resp. $\mathcal{R}(O^*(2n))$) which can be realized as quotients of $\mathcal{F}_{p,q,n}$. We define the set $\mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$ in the same way. Howe [3] proved the following theorem.

Theorem 3.1. *The set $\mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$ defines a bijection between $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$ and $\mathcal{R}(O^*(2n), \omega_{p,q,n})$. For $\pi \in \mathcal{R}(Sp(p, q), \omega_{p,q,n})$ and $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$, they correspond to each other in this bijection if and only if $\pi \otimes \pi' \in \mathcal{R}(Sp(p, q) \times O^*(2n), \omega_{p,q,n})$.*

We call this bijection the local theta correspondence or the Howe correspondence. We denote by $\theta_{p,q}$ the bijection from the set $\mathcal{R}(O^*(2n), \omega_{p,q,n})$ to the set $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$ and by θ_n the bijection in the other direction. We say $\theta_{p,q}(\pi') = 0$ for $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$ if $\pi' \notin \mathcal{R}(O^*(2n), \omega_{p,q,n})$. Similarly, we say $\theta_n(\pi) = 0$ for $\pi \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$ if $\pi \notin \mathcal{R}(Sp(p, q), \omega_{p,q,n})$. Formally, we also define the theta correspondence for reductive dual pairs $(Sp(p, q), O^*(0))$ and $(Sp(0, 0), O^*(2n))$ while only trivial representations occur in the correspondence.

In [2], the Fock space $\mathcal{F}_{p,q,n}$ can be realized on a complex polynomial space with $2n(p+q)$ variables and the action of the two-fold metaplectic cover \widetilde{U} keeps the degree of polynomials invariant. This allow us to associate to each $Sp(p) \times Sp(q)$ -type σ (resp. $U(n)$ -type σ') occurring in $\mathcal{F}_{p,q,n}$ a degree which is the minimal degree of polynomials in the σ -isotypic subspace (resp. σ' -isotypic subspace) of $\mathcal{F}_{p,q,n}$. For $\pi \in \mathcal{R}(Sp(p, q), \omega_{p,q,n})$, a $Sp(p) \times Sp(q)$ -type σ is called a lowest degree $Sp(p) \times Sp(q)$ -type of π if and only if it is of minimal degree among all $Sp(p) \times Sp(q)$ -types occurring in π . Similarly, we define the lowest degree $U(n)$ -types for $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$. Howe [3] proved that there exists a $(Sp(p) \times Sp(q)) \times U(n)$ -invariant subspace $\mathcal{H}_{p,q,n}$ of $\mathcal{F}_{p,q,n}$ called the space of joint harmonics with the following properties.

Theorem 3.2. *There is a one-one correspondence of $Sp(p) \times Sp(q)$ -types and $U(n)$ -types on the joint harmonics $\mathcal{H}_{p,q,n}$ with the following properties. Suppose $\pi \in \mathcal{R}(Sp(p, q), \omega_{p,q,n})$ and $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$ correspond to each other in the theta correspondence for $(Sp(p, q), O^*(2n))$. Let σ be a lowest degree $Sp(p) \times Sp(q)$ -type of π . Then σ occurs in the joint harmonics $\mathcal{H}_{p,q,n}$ and corresponds to a $U(n)$ -type σ' . Furthermore, σ' is a lowest degree $U(n)$ -type of π' . The statement of this theorem is also true with the roles of π and π' (also of σ and σ') reversed.*

Let σ be a $Sp(p) \times Sp(q)$ -type and let σ' be a $U(n)$ -type. Write

$$\begin{cases} \sigma = (a_1, a_2, \dots, a_r, 0, \dots, 0; b_1, b_2, \dots, b_s, 0, \dots, 0) & r \leq p, s \leq q, \\ \sigma' = (a'_1, a'_2, \dots, a'_r, 0, \dots, 0, -b'_s, \dots, -b'_2, -b'_1) + & \\ (p-q, \dots, p-q) & r' + s' \leq n, \end{cases} \quad (38)$$

where $a_1 \geq \dots \geq a_r > 0$, $b_1 \geq \dots \geq b_s > 0$, and similarly for the a'_i and b'_j . We have the following lemma (Lemma 3.4 of [11]).

Lemma 3.3. (1) Let σ be a $Sp(p) \times Sp(q)$ -type as in (38). Then σ occurs in $\mathcal{F}_{p,q,n}$ if and only if $r, s \leq n$. Let σ' be a $U(n)$ -type as in (38). Then σ' occurs in $\mathcal{F}_{p,q,n}$ if and only if $r' \leq 2p, s' \leq 2q$. If the conditions are satisfied, then

$$\deg(\sigma) = a_1 + \cdots + a_r + b_1 + \cdots + b_s, \quad (39)$$

$$\deg(\sigma') = a'_1 + \cdots + a'_{r'} + b'_1 + \cdots + b'_{s'}. \quad (40)$$

(2) Let σ be a $Sp(p) \times Sp(q)$ -type as in (38). Then σ occurs in the joint harmonics if and only if $r + s \leq n$. Let σ' be a $U(n)$ -type as in (38). Then σ' occurs in the joint harmonics if and only if $r' \leq p, s' \leq q$. Furthermore, σ corresponds to σ' in the joint harmonics if and only if $r = r', s = s', a_i = a'_i$ and $b_j = b'_j$.

Let H be a real reductive group with a maximal compact subgroup K_H . For an admissible representation π of H , we denote by π^\vee its contragredient representation. We have the following lemma (see Lemma 3.32 of [11]).

Lemma 3.4. Let π' be an irreducible admissible representation of $O^*(2n)$. Then $(\theta_{p,q}(\pi'))^\vee = \theta_{q,p}((\pi')^\vee)$.

Notice that if π is an irreducible admissible representation of $Sp(p, q)$, $\pi = \pi^\vee$. By Lemma 3.4, we reduce the problem to calculating $\theta_{p,q}$ ($p \geq q$) explicitly. The following theorem is on the correspondence of infinitesimal characters (see Theorem 1.19 of [17]).

Theorem 3.5. Let p, q, n be non-negative integers such that $p + q \geq n$. Suppose $\pi \leftrightarrow \pi'$ for the reductive dual pair $(Sp(p, q), O^*(2n))$. If the infinitesimal character of π' is $(\lambda'_1, \lambda'_2, \dots, \lambda'_n)$, then the infinitesimal character of π is $(\lambda'_1, \lambda'_2, \dots, \lambda'_n, p + q - n, \dots, 2, 1)$.

To reduce the problem of explicit calculation, we introduce the stable range theorem. In [16], the authors proved the stable range theorem.

Theorem 3.6. Let π' be an irreducible admissible representation of $O^*(2n)$. Then $\theta_{p,q}(\pi')$ is nonzero if $\min\{p, q\} \geq n$.

For $\pi \in \mathcal{R}(Sp(p, q), \omega_{p,q,n})$, we denote by $\mathcal{A}_{p,q,n}(\pi)$ the set of lowest degree $Sp(p) \times Sp(q)$ -types occurring in π for the local theta correspondence for $(Sp(p, q), O^*(2n))$. Similarly, for $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$, we also define the set $\mathcal{A}_{p,q,n}(\pi')$. We state the following lemma without proof (see Lemma 5.1.1 of [14]).

Lemma 3.7. Let $X(r, \lambda, \Psi, \mu, \nu)$ be the standard module of $Sp(p, q)$ as in (15) and let σ be a $Sp(p) \times Sp(q)$ -type occurring in $X(r, \lambda, \Psi, \mu, \nu)$. Then σ is of the form

$$\sigma = \Lambda + \sum n_\alpha \alpha,$$

where Λ is a lowest K -type of $X(P, r, \lambda, \Psi, \mu, \nu)$, the sum runs over roots in $\Delta(\mathfrak{l} : \mathfrak{t}) \cup \Delta(\mathfrak{u} \cap \mathfrak{p})$ and $n_\alpha \geq 0$ for all α . Here \mathfrak{l} and \mathfrak{u} are those in §2.3.

This lemma is due to the standard theory in [5]. By this lemma, we prove the following theorem.

Theorem 3.8. Let $\pi = \pi(r, \lambda, \Psi, \mu, \nu)$ be a representation in $\mathcal{R}(Sp(p, q), \omega_{p,q,n})$. Then $\mathcal{A}(\pi) \subseteq \mathcal{A}_{p,q,n}(\pi)$.

PROOF. By Theorem 2.2 and Lemma 3.3, it is not difficult to check that the degrees of all lowest $S p(p) \times S p(q)$ -types of π equal to each other. By Lemma 3.7, any $S p(p) \times S p(q)$ -type σ occurring in $\pi(r, \lambda, \Psi, \mu, \nu)$ is of the form $\Lambda + \Sigma n_\alpha \alpha$, where Λ is a lowest $S p(p) \times S p(q)$ -type of π . The roots in $\Delta(\mathfrak{l} : \mathfrak{t}) \cup \Delta(\mathfrak{u} \cap \mathfrak{p})$ are of the form

$$\begin{aligned} \pm(e_i - e_j) \quad & 1 \leq i < j \leq p, \\ \pm(f_i - f_j) \quad & 1 \leq i < j \leq q, \\ \pm(e_i - f_j) \quad & 1 \leq i \leq p, 1 \leq j \leq q, \\ e_i + f_j \quad & 1 \leq i \leq p, 1 \leq j \leq q. \end{aligned} \tag{41}$$

Let $\Sigma n_\alpha \alpha = (x_1, \dots, x_p; y_1, \dots, y_q)$. By (39), $\deg(\sigma) = \deg(\Lambda) + \sum_{i=1}^p x_i + \sum_{j=1}^q y_j$. For each α of the form in (41), the summation of its all coordinates is non-negative. Then $\deg(\sigma) \geq \deg(\Lambda)$ and Λ is a lowest degree $S p(p) \times S p(q)$ -type of π . Furthermore, all lowest $S p(p) \times S p(q)$ -types are of minimal degree. We finish the proof.

For $\pi' \in \mathcal{R}(O^*(2n), \omega_{p,q,n})$, $\mathcal{A}(\pi')$ is not necessarily contained in $\mathcal{A}_{p,q,n}(\pi')$ since the degree formula (40) depends on the choice of (p, q) .

4. Parabolic Induction Principle

In this chapter, we briefly review the mechanism of parabolic induction principle. The induction principle is due to Kudla [6]. The proof of parabolic induction principle is based on the explicit realization of the Schrodinger Model (see [18]). We mainly follow the content in §4 of [11]. The detailed proof of induction principle can also be founded in §3 of [1] and §4 of [14]. Using induction principle, we prove the Going-Up theorem which is useful in explicit calculation.

Let $\mathbf{D} = \mathbb{H}$ with the standard involution as in §2.

Let V (resp. V') be a right (resp. left) \mathbf{D} space equipped with a Hermitian form $(,)$ (resp. skew-Hermitian form \langle, \rangle). We define $W = V \otimes_{\mathbf{D}} V'$ to be a real symplectic space with the symplectic form $\langle, \rangle = \text{tr}_{\mathbf{D}/\mathbb{R}}((,) \otimes \overline{\langle, \rangle})$, where tr is the reduced trace on \mathbb{H} such that $\text{tr}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = 2a$. We denote by $G = G(V)$ (resp. $G' = G(V')$) the isometry group of $(V, (,))$ (resp. (V', \langle, \rangle)). As in §3, $(G(V), G(V'))$ can be embedded into $S p(W)$ and is a reductive dual pair in this symplectic group.

Suppose that

$$V = V_+ \oplus V_0 \oplus V_-, \tag{42}$$

$$V' = V'_+ \oplus V'_0 \oplus V'_-, \tag{43}$$

where V_+, V_- (resp. V'_+, V'_-) are totally isotropic subspaces and are dual to each other with respect to $(,)$ (resp. \langle, \rangle). Denote $d = \dim_{\mathbb{R}} \mathbb{H} = 4$ and $d_0 = \dim_{\mathbb{R}} \{t \in \mathbb{H} | \bar{t} = -t\} = 3$. Set $m_0 = \dim_{\mathbb{H}} V_0$, $n_0 = \dim_{\mathbb{H}} V'_0$, $k = \dim_{\mathbb{H}} V_+$ and $l = \dim_{\mathbb{H}} V'_+$.

Let $W_0 = V_0 \otimes_{\mathbb{H}} V'_0$ and let $W_0 = X_0 \oplus Y_0$ is a complete polarization. Then $W = X \oplus Y$ is a complete polarization of W , where

$$X = (V \otimes_{\mathbb{H}} V'_+) \oplus (V_+ \otimes_{\mathbb{H}} V'_0) \oplus (X_0), \tag{44}$$

$$Y = (V \otimes_{\mathbb{H}} V'_-) \oplus (V_- \otimes_{\mathbb{H}} V'_0) \oplus (Y_0). \tag{45}$$

Let $P_V = P(V_+)$ be the stabilizer group of V_+ in $G(V)$ and let $P_{V'} = P(V'_+)$ be the stabilizer group of V'_+ in $G(V')$. Then $P_V = L_V N_V$ and $P_{V'} = L_{V'} N_{V'}$, where $L_V \cong G(V_0) \times GL(k, \mathbb{H})$ as in (14) and $L_{V'} \cong G(V'_0) \times GL(l, \mathbb{H})$ as in (27).

Let $W_L = (V_- \otimes_{\mathbb{H}} V'_+) \oplus (V_+ \otimes_{\mathbb{H}} V'_-) \oplus (V_0 \otimes_{\mathbb{H}} V'_0)$. Then W_L is a real symplectic subspace of W . Let $W_L = X_L \oplus Y_L$ be a complete polarization of W_L , where

$$X_L = (V_- \otimes_{\mathbb{H}} V'_+) \oplus X_0, \quad (46)$$

$$Y_L = (V_+ \otimes_{\mathbb{H}} V'_-) \oplus Y_0. \quad (47)$$

Then $(L_V, L_{V'})$ is a reductive dual pair in $Sp(W_L)$. In fact, $(G(V_0), G(V'_0))$ is a reductive dual pair of type I and $(GL(k, \mathbb{H}), GL(l, \mathbb{H}))$ is a reductive dual pair of type II (c.f. §6 of [11]).

For a fixed additive character $\psi(t) = e^{2\pi i t}$, we denote by ω the oscillator representation attached to the dual pair $(G(V), G(V')) \subseteq Sp(W)$ and by ω_L the oscillator representation attached to the dual pair $(L_V, L_{V'}) \subseteq Sp(W_L)$. The oscillator representation ω may be realized on the Schwartz space $S(Y)$ and the oscillator representation ω_L may be realized on the Schwartz space $S(Y_L)$ (see [18]). Let

$$\rho : S(Y) \rightarrow S(Y_L) \quad (48)$$

be the restriction map. Obviously ρ is surjective.

By the homomorphism (4), the group $GL(k, \mathbb{H})$ can be embedded into the group $GL(2k, \mathbb{C})$. For every $h \in GL(k, \mathbb{H})$, let $\det(h)$ be the usual determinant of h realized as an element in $GL(2k, \mathbb{C})$. Similarly, we define the determinant of $GL(l, \mathbb{H})$. Let ξ be a character of $P_V \times P_{V'}$:

$$\xi((g_0 h n, g'_0 h' n')) = \det(h)^{n_0+l} \det(h')^{m_0+k}, \quad (49)$$

where $(g_0, g'_0) \in (G(V_0), G(V'_0))$, $(h, h') \in (GL(k, \mathbb{H}), GL(l, \mathbb{H}))$ and $(n, n') \in (N_V, N_{V'})$. Since N_V (resp. $N_{V'}$) is normal in P_V (resp. $P_{V'}$), the character is well-defined. Then we have the following theorem (Proposition 4.14 in [11]).

Theorem 4.1. *Let ξ be the character as in (49). The restriction map ρ in (48) is a $P_V \times P_{V'}$ -equivalent map*

$$\omega \rightarrow \omega_L \otimes \xi, \quad (50)$$

where $N_V \times N_{V'}$ acts trivially on ω_L .

The detail of proof can be founded in §4 of [14]. Let χ_V and $\chi_{V'}$ be the characters of $GL(k, \mathbb{H})$ and $GL(l, \mathbb{H})$ given by

$$\chi_V(h) = \det(h)^{\frac{1}{4}((n_0+l-m_0-k+1)d-2d_0)}, \quad \forall h \in GL(k, \mathbb{H}), \quad (51)$$

$$\chi_{V'}(h') = \det(h')^{\frac{1}{4}((m_0+k-n_0-l-1)d+2d_0)}, \quad \forall h' \in GL(l, \mathbb{H}). \quad (52)$$

Using Frobenius Reciprocity, we state the following theorem on Induction Principle (Theorem 4.20 of [11], also see Theorem 5.10 of [15]).

Theorem 4.2. *Let $\pi \in \mathcal{R}(G(V_0))$, $\pi' \in \mathcal{R}(G(V'_0))$, $\tau \in \mathcal{R}(GL(k, \mathbb{H}))$ and $\tau' \in \mathcal{R}(GL(l, \mathbb{H}))$. Suppose that π corresponds to π' for the real reductive dual pair $(G(V_0), G(V'_0))$ and τ corresponds to τ' for the real reductive dual pair $(GL(k, \mathbb{H}), GL(l, \mathbb{H}))$. There is a nonzero $G(V) \times G(V')$ map (on the level of Harish-Chandra module)*

$$\Phi : \omega \rightarrow \text{Ind}_{P_V}^{G(V)}(\pi \otimes \tau \otimes \chi_V \otimes \mathbb{1}) \otimes \text{Ind}_{P_{V'}}^{G(V')}(\pi' \otimes \tau' \otimes \chi_{V'} \otimes \mathbb{1}). \quad (53)$$

To be more precise, we state the following theorem (Theorem 4.24 of [11]). The details of proof can be founded in §3 of [1].

Theorem 4.3. *Let Φ be as in Theorem 4.2, Denote $G = G(V)$, $G' = G(V')$, $L = L_V$ and $L' = L_{V'}$, and let K and K' be the maximal compact subgroups of G and G' , respectively. Suppose σ is a K -type and κ is a $K \cap L$ -type such that the following conditions are satisfied:*

- (1) κ occurs and is of minimal degree in $\pi \otimes \tau$.
- (2) σ occurs and is of minimal degree and of multiplicity one in $\text{Ind}_{P_V}^{G(V)}(\pi \otimes \tau \otimes \chi_V \otimes \mathbb{1})$.
- (3) σ and κ have the same degree, and the restriction of σ to $K \cap L$ contains κ .
- (4) There exist characters α and α' of L and L' which are trivial on $K \cap L$ and $K' \cap L'$, such that $\pi \otimes \tau \otimes \alpha$ and $\pi' \otimes \tau' \otimes \alpha'$ correspond to each other for the reductive dual pair (L, L') , and $\text{Ind}_{P_V}^{G(V)}(\pi \otimes \tau \otimes \alpha \otimes \chi_V \otimes \mathbb{1})$ is irreducible.

Let σ' be the K' type which corresponds to σ in the joint harmonics space. Then $\sigma \otimes \sigma'$ is in the image of Φ .

The statement of this theorem is also true if we exchange the role of V and V' .

For further application, we need to know something about the theta correspondence for the reductive dual pairs $(GL(n, \mathbb{H}), GL(m, \mathbb{H}))$. For positive integers n, m , $(GL(n, \mathbb{H}), GL(m, \mathbb{H}))$ is a reductive dual pair in $Sp(8mn, \mathbb{R})$ (see §6 of [11]). The maximal compact subgroups of $GL(n, \mathbb{H})$ and $GL(m, \mathbb{H})$ are isomorphic to $Sp(n)$ and $Sp(m)$, respectively. We state the following two propositions (Proposition 4.25 and 4.27 of [11]).

Proposition 4.4. *Let $n \leq m$. The correspondence of $Sp(n)$ -types and $Sp(m)$ -types in the joint harmonics for the reductive dual pair $(GL(n, \mathbb{H}), GL(m, \mathbb{H}))$ is given as follow. If $\sigma = (a_1, \dots, a_n)$ is a $Sp(n)$ -type, then σ occurs in the joint harmonics and corresponds to the $Sp(m)$ -type $\sigma' = (a_1, \dots, a_n, 0, \dots, 0)$. The degree of σ is $\sum_{i=1}^n a_i$.*

Proposition 4.5. *Let $\tau = \tau(\mu, \nu)$ be an irreducible admissible representation of $GL(1, \mathbb{H})$ as in §2.1. Then τ corresponds to τ^\vee for the reductive dual pair $(GL(1, \mathbb{H}), GL(1, \mathbb{H}))$, where $\tau^\vee = \tau(\mu, -\nu)$.*

Using Induction Principle, we prove the following theorem.

Theorem 4.6. *Let π' be an irreducible admissible representation of $O^*(2n)$. Suppose $\theta_{p,q}(\pi')$ is nonzero and $\theta_{p,q}(\pi') = \pi(r, \lambda, \Psi, \mu, \nu)$. Let s be a non-negative integer. Then*

$$\theta_{p+s,q+s}(\pi') = \pi(r+s, \lambda, \Psi, \mu^s, \nu^s),$$

where $\mu^s = (\mu, \underbrace{1, \dots, 1}_s)$ and

$$\nu^s = (\nu, 2p+2q-2n+3, 2p+2q-2n+7, \dots, 2p+2q-2n+4s-1).$$

PROOF. We use induction on s . Let $\pi_s = \pi(r+s, \lambda, \Psi, \mu^s, \nu^s)$. By Theorem 3.2 and Theorem 3.8, there exists a lowest $Sp(p) \times Sp(q)$ -type η_0 of $\pi_0 = \theta_{p,q}(\pi')$ such that η_0 corresponds to a lowest degree $U(n)$ -type η' of π' in $\mathcal{H}_{p,q,n}$. Assume $\theta_{p+s-1,q+s-1}(\pi') = \pi_{s-1} = \pi(r+s-1, \lambda, \Psi, \mu^{s-1}, \nu^{s-1})$. Take $G(V) = Sp(p+s, q+s)$, $G(V') = O^*(2n)$, $L = Sp(p+s-1, q+s-1) \times GL(1, \mathbb{H})$ and $L' = O^*(2n)$. By Theorem 4.2, there is a nonzero $Sp(p+s, q+s) \times O^*(2n)$ map

$$\omega_{p+s,q+s,n} \mapsto \text{Ind}_P^{Sp(p+s,q+s)}(\pi_{s-1} \otimes \chi_s \otimes \mathbb{1}) \otimes \pi',$$

where $P = LN$ with $L = Sp(p + s - 1, q + s - 1) \cdot GL(1, \mathbb{H})$, and χ_s is a character of $GL(1, \mathbb{H})$ such that

$$\chi_s(h) = \det(h)^{n-p-q-2s+\frac{1}{2}}$$

for each $h \in GL(1, \mathbb{H})$. Then $\theta_{p+s, q+s}(\pi')$ is a constituent of the induced representation $Ind_P^{Sp(p+s, q+s)}(\pi_{s-1} \otimes \chi_s \otimes \mathbb{1})$. Let P_{s-1} be a cuspidal subgroup of $Sp(p + s - 1, q + s - 1)$ with Levi component $L_{s-1} \cong Sp(p - r, q - r) \times GL(1, \mathbb{H})^{r+s-1}$. Then π_{s-1} is a constituent of the standard module $X(r + s - 1, \lambda, \Psi, \mu^{s-1}, \nu^{s-1})$. Since $P_{s-1} \cdot GL(1, \mathbb{H})$ is a parabolic subgroup of L , then $P_s = P_{s-1} \cdot GL(1, \mathbb{H}) \cdot N$ is a parabolic subgroup of $Sp(p + s, q + s)$. By the double induction formula in §7.2 of [4], the induced representation $Ind_P^{Sp(p+s, q+s)}(\pi_{s-1} \otimes \chi_s \otimes \mathbb{1})$ is a sub-quotient of the standard module $X(r + s, \lambda, \Psi, \mu_s, \nu_s)$, where $\mu_s = (\mu^{s-1}, 1)$ and $\nu_s = (\nu^{s-1}, 2n - 2p - 2q - 4s + 1)$. Furthermore, $\theta_{p+s, q+s}(\pi')$ is a constituent of this standard module. Suppose $\eta_0 = (a_1, \dots, a_p; b_1, \dots, b_q)$. Let η_s be a $Sp(p + s) \times Sp(q + s)$ -type such that

$$\eta_s = (a_1, \dots, a_p, \overbrace{0, \dots, 0}^s; b_1, \dots, b_q, \overbrace{0, \dots, 0}^s).$$

Since η_0 corresponds to η' in the joint harmonics $\mathcal{H}_{p,q,n}$, then η_s corresponds to η' in the joint harmonics $\mathcal{H}_{p+s, q+s, n}$ by Lemma 3.3. By Theorem 2.2 (also see the proof of Theorem 5.8 in [11]) and Remark 2.3, η_s is the lowest $Sp(p + s) \times Sp(q + s)$ -type of the standard module $X(r + s, \lambda, \Psi, \mu_s, \nu_s)$ with multiplicity one. By Theorem 3.2, π' corresponds to $\pi(r + s, \lambda, \Psi, \mu_s, \nu_s)$, the unique constituent of $X(r + s, \lambda, \Psi, \mu_s, \nu_s)$ containing η_s . By Theorem 2.1, we finish the proof.

5. Explicit Theta Correspondence for $(Sp(p, q), O^*(2))$

In this chapter, we briefly state the explicit theta correspondence for reductive dual pairs $(Sp(p, q), O^*(2))$. This work is done by R. Howe and J.-S. Li. Their strategy is suggestive for calculating explicit theta correspondence for reductive dual pairs $(Sp(p, q), O^*(4))$.

By Lemma 3.4, Theorem 3.6 and Theorem 4.6, the problem is reduced to determination of $\theta_{p,q}$ for $p \geq q$ and $q \leq 1$. Irreducible admissible representations of $O^*(2) = U(1)$ are $\chi_k : t \mapsto t^k$ with $k \in \mathbb{Z}$.

The following is a list of systems of positive roots which will be mentioned in this chapter:

- (1) For each integer $p \geq 0$, we denote by Ψ_1 the system of positive roots for the compact group $Sp(p)$ such that $(p, p - 1, \dots, 1)$ is dominant with respect to Ψ_1 ;
- (2) For each integer $p \geq 1$, we denote by Ψ_2 the system of positive roots for the group $Sp(p, 1)$ such that $(p, p - 1, \dots, 1; p + 1)$ is dominant with respect to Ψ_2 ;
- (3) For each integer $p \geq 1$, we denote by Ψ_3 the system of positive roots for the group $Sp(p, 1)$ such that $(p + 1, p, \dots, 2; 1)$ is dominant with respect to Ψ_3 .

For $p \geq 1$, we denote by $\mathbf{A}_{p,1}$ the set of elements in $\mathcal{R}(Sp(p, 1))$ whose lowest $Sp(p) \times Sp(1)$ -types are all of the form $(a_1, 0, \dots, 0; b_1)$. In the appendix, we list all infinitesimal equivalence classes in $\mathbf{A}_{p,1}$ in terms of Langlands parameters.

Theorem 5.1. *The following is a list of all explicit theta $(p, 0)$ -lifts on $\mathcal{R}(O^*(2))$ for $p \geq 0$:*

- (1) If $p = 0$, then

$$\theta_{0,0}(\chi_k) = \begin{cases} \pi(0, \emptyset), & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

(2) If $p \geq 1$, then

$$\theta_{p,0}(\chi_k) = \begin{cases} 0, & \text{if } k < p, \\ \pi((k, p-1, p-2, \dots, 1), \Psi_1), & \text{if } k \geq p. \end{cases}$$

PROOF. First assume $p = 0$. Then $Sp(0, 0)$ is the trivial group and the explicit theta correspondence $\theta_{0,0}$ is obvious. The trivial representation of $Sp(0, 0)$ in terms of Langlands parameters is $\pi(0, \emptyset)$. Next assume $p > 0$. By Lemma 3.3, the $U(1)$ -type (k) does not occur in the Fock space $\mathcal{F}_{p,0,1}$ if $k < p$. Then $\theta_{p,0}(\chi_k) = 0$ if $k < p$. If $k \geq p$, by Proposition 6.1 and Theorem 6.2 of [10] (also see Proposition 6.1 of [25]), χ_k can be theta lifted to a unitary representation with nonzero cohomology. By Theorem 3.2, Lemma 3.3, Theorem 3.5 and Theorem 3.8, $\theta_{p,0}(\chi_k)$ has a lowest $Sp(p)$ -type $(k-p, 0, \dots, 0)$ and its infinitesimal character is $(k, p-1, p-2, \dots, 1)$. It is easy to see that

$$\theta_{p,0} = \pi((k, p-1, p-2, \dots, 1), \Psi_1)$$

if $k \geq p$.

Theorem 5.2. *The following is a list of all explicit theta $(p, 1)$ -lifts on $\mathcal{R}(O^*(2))$ for $p \geq 1$:*

(1) If $p = 1$, then

$$\theta_{1,1}(\chi_k) = \begin{cases} \pi((k; 1), \Psi_3), & \text{if } k > 0, \\ \pi((1; -k), \Psi_2), & \text{if } k < 0, \\ \pi(1, 0, \emptyset, 1, 1), & \text{if } k = 0. \end{cases}$$

(2) If $p > 1$ and $k \geq p-1$, then

$$\theta_{p,1}(\chi_k) = \pi(1, (k, p-2, p-3, \dots, 1), \Psi_1, 1, 2p-1).$$

(3) If $p > 1$ and $-p < k < p-1$, then

$$\theta_{p,1}(\chi_k) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, p-k, p+k).$$

(4) If $p > 1$ and $k \leq -p$, then

$$\theta_{p,1}(\chi_k) = \pi((p, p-1, \dots, 1; -k), \Psi_2).$$

PROOF. By Theorem 3.6, $\theta_{p,1}(\chi_k)$ is nonzero for each $k \in \mathbb{Z}$. If $k \geq p-1$, by Theorem 3.2, Lemma 3.3, Theorem 3.5 and Theorem 3.8, we know that $\theta_{p,1}(\chi_k)$ has a unique lowest $Sp(p) \times Sp(1)$ -type $(k-p+1, 0, \dots, 0; 0)$ and its infinitesimal character is $(k, p, p-1, \dots, 1)$. By the appendix, there is a unique element in $\mathbf{A}_{p,1}$ satisfying these conditions. Similarly, if $k < p-1$, $\theta_{p,1}(\chi_k)$ has a unique lowest $Sp(p) \times Sp(1)$ -type $(0, 0, \dots, 0; p-1-k)$ and its infinitesimal character is $(k, p, p-1, \dots, 1)$. By the appendix, there is a unique element in $\mathbf{A}_{p,1}$ satisfying these conditions. Then all desired theta $(p, 1)$ -lifts in this theorem are determined in terms of Langlands parameters.

6. Explicit Theta Correspondence For $(Sp(p, q), O^*(4))$

In this chapter, we determine theta correspondence for reductive dual pairs $(Sp(p, q), O^*(4))$ explicitly in terms of Langlands parameters. By Lemma 3.4, Theorem 3.6 and Theorem 4.6, the problem is reduced to determination of $\theta_{p,q}$ explicitly for $p \geq q$ and $q \leq 2$. Here we list all infinitesimal equivalence classes in $\mathcal{R}(O^*(4))$ as in Theorem 2.7:

- (1) P_{λ_1, λ_2} with $\lambda_1, \lambda_2 \notin \mathbb{Z}$, $\operatorname{Re}(\lambda_1 + \lambda_2) \geq 0$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (2) D_{λ_1, λ_2} with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (3) $\overline{D}_{\lambda_1, \lambda_2}$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\leq 0}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$;
- (4) F_{λ_1, λ_2} with $\lambda_1, \lambda_2 \in \mathbb{Z}$, $\lambda_1 + \lambda_2 \in \mathbb{Z}_{\geq 1}$ and $\lambda_1 - \lambda_2 \in \mathbb{Z}_{\geq 1}$.

Let condition (A) and (B) be those of §2.1. Let Ψ_1, Ψ_2 and Ψ_3 be those in §5. The following is a complementary list of the systems of positive roots which will be mentioned in this chapter:

- (1) For $p \geq 1$, we denote by Ψ_4 the system of positive roots of the group $Sp(p, 1)$ such that $(p+1, p-1, p-2, \dots, 1; p)$ is dominant with respect to Ψ_4 ;
- (2) For $p \geq 2$, we denote by Ψ_5 the system of positive roots of the group $Sp(p, 2)$ such that $(p, p-1, \dots, 1; p+2, p+1)$ is dominant with respect to Ψ_5 ;
- (3) For $p \geq 2$, we denote by Ψ_6 the system of positive roots of the group $Sp(p, 2)$ such that $(p+2, p+1, \dots, 3; 2, 1)$ is dominant with respect to Ψ_6 .

6.1. Explicit Theta $(p, 0)$ -Lifting for $O^*(4)$

Theorem 6.1. *The following is a list of all nonzero theta $(p, 0)$ -lifts on $\mathcal{R}(O^*(4))$ for $p \geq 0$:*

- (1) If $p = 0$, then

$$\theta_{0,0}(F_{1,0}) = \pi(0, \emptyset).$$

- (2) If $p = 1$, then

$$\theta_{1,0}(D_{\lambda_1,0}) = \pi((\lambda_1), \Psi_1).$$

- (3) If $p \geq 2$ and $\lambda_2 \geq p-1$, then

$$\theta_{p,0}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, \lambda_2, p-2, p-3, \dots, 1), \Psi_1).$$

PROOF. The theta correspondence for $(Sp(0, 0), O^*(4))$ is trivial. The trivial representation of $Sp(0, 0)$ is $\pi(0, \emptyset)$ and the trivial representation of $O^*(4)$ is $F_{1,0}$. By Theorem 5.1 of [11], we determine $\theta_{1,0}$ directly. Assume $p \geq 2$. For $\pi' \in \mathcal{R}(O^*(4))$, $\theta_{p,0}(\pi')$ is nonzero only if all $U(2)$ -types in $\mathcal{K}(\pi')$ occur in $\mathcal{F}_{p,0,2}$. By Theorem 2.7 and Lemma 3.3, $\theta_{p,0}(\pi')$ is nonzero only if π' is of the form D_{λ_1, λ_2} with $\lambda_2 \geq p-1$. If $\lambda_2 \geq p-1$, by Proposition 6.1 and Theorem 6.2 of [10],

$$\theta_{p,0}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, \lambda_2, p-2, p-3, \dots, 1), \Psi_1).$$

6.2. Explicit Theta $(p, 1)$ -Lifting for $O^*(4)$

Assume $p \geq 1$. Let $G = Sp(p, 1)$ and let $G' = O^*(4)$. As in §2.4, $G' = (G'_1 \times G'_2)/\{\pm I\}$ with $G'_1 = SU(2)$ and $G'_2 = SL(2, \mathbb{R})$. Let P'_2 be the upper triangle matrix subgroups of G'_2 and let $P' = (G'_1 \times P'_2)/\{\pm I\}$. Let P be a proper cuspidal parabolic subgroup of G . The Levi component L of P is isomorphic to $Sp(p-1) \times GL(1, \mathbb{H})$. For $\mu \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{C}$, by Theorem 4.2 and Proposition 4.5, we have a nonzero $Sp(p, 1) \times O^*(4)$ -projection (on the level of Harish-Chandra module)

$$\Phi : \omega_{p,1,2} \mapsto \operatorname{Ind}_P^{Sp(p,1)}(\tau(\mu, -\nu) \otimes \mathbb{1}) \otimes \operatorname{Ind}_{P'}^{O^*(4)}(\tau(\mu, \nu) \otimes \mathbb{1}), \quad (54)$$

where $\tau(\mu, \nu)$ is the irreducible representation of $GL(1, \mathbb{H})$ as in §2.1. For convenience, we denote the first standard module by $I(\mu, -\nu)$ and the second standard module by $I'(\mu, \nu)$. We calculate the explicit theta $(p, 1)$ -lifts of irreducible principal series first.

Theorem 6.2. *The following is a list of explicit theta $(p, 1)$ -lifts of irreducible principal series:*

$$\theta_{p,1}(P_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

PROOF. By Theorem 2.7, the representation P_{λ_1, λ_2} is the standard module $I'(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$. By (54), there is a nonzero $Sp(p, 1) \times O^*(4)$ -projection Φ from $\omega_{p,1,2}$ to $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2)) \otimes I'(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$. Then we know that $\theta_{p,1}(P_{\lambda_1, \lambda_2})$ is a constituent of $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2))$. By Theorem 1.1 of [20], $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2))$ is irreducible (also see Theorem 5.2.2 of [12]) and $\theta_{p,1}(P_{\lambda_1, \lambda_2})$ is just the standard module $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2))$. By Theorem 2.1, $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2))$ is just $\pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$.

Next we calculate the explicit theta $(p, 1)$ -lifts of irreducible finite dimensional representations F_{λ_1, λ_2} .

Theorem 6.3. *The following is a list of explicit theta $(p, 1)$ -lifts of irreducible finite dimensional representations of $O^*(4)$ for $p \geq 1$:*

(1) *If $\lambda_1 \geq p$, then*

$$\theta_{p,1}(F_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

(2) *If $\lambda_1 < p$, then*

$$\theta_{p,1}(F_{\lambda_1, \lambda_2}) = 0.$$

PROOF. First assume $\lambda_1 < p$. By Theorem 2.7 and Lemma 3.3, every $U(2)$ -type in $\mathcal{K}(F_{\lambda_1, \lambda_2})$ does not occur in the joint harmonics $\mathcal{H}_{p,1,2}$. Then $\theta_{p,1}(F_{\lambda_1, \lambda_2})$ is zero by Theorem 3.2.

Next assume $\lambda_1 \geq p$. Let Φ be the nonzero projection from $\omega_{p,1,2}$ to $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2)) \otimes I'(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$ as in (54). Denote the first standard module by I and the second standard module by I' . If $\lambda_2 \leq 1 - p$, by Theorem 2.7 and Lemma 3.3, we know that the $U(2)$ -type $\eta' = (-\lambda_2, 1 - \lambda_1)$ occurs in F_{λ_1, λ_2} (also in I') with multiplicity one and is of the minimal degree $\lambda_1 - \lambda_2 - 1$. The restriction of η' to $SU(2)$ is the $SU(2)$ -type $(\lambda_1 - \lambda_2 - 1)$ and has degree $\lambda_1 - \lambda_2 - 1$ by Proposition 4.4. Then η' satisfy the condition (1), (2) and (3) of Theorem 4.3, and the condition (4) is also satisfied by Theorem 1.1 of [20]. By Theorem 4.3, we know that $\eta \otimes \eta'$ occurs in $\text{Im}\Phi$, where η corresponds to η' in $\mathcal{H}_{p,1,2}$. By Theorem 2.5, F_{λ_1, λ_2} can be regarded as a quotient of I' such that $F_{\lambda_1, \lambda_2} = I'/M'$. Denote $\mathcal{M} = I \otimes M'$. Then $\text{Im}\Phi/(\text{Im}\Phi \cap \mathcal{M})$ is a nonzero submodule of $I \otimes F_{\lambda_1, \lambda_2}$ ($\eta \otimes \eta'$ does not lie in \mathcal{M}). Since $\text{Im}\Phi/(\text{Im}\Phi \cap \mathcal{M})$ is a $Sp(p, 1) \times O^*(4)$ -quotient of $\omega_{p,1,2}$, we know that $\theta_{p,1}(F_{\lambda_1, \lambda_2})$ is a constituent of I . If $\lambda_2 > 1 - p$, take $\eta' = (p-1, p - \lambda_1 + \lambda_2)$. Similarly, we prove that $\theta_{p,1}(F_{\lambda_1, \lambda_2})$ is a constituent of I .

By Theorem 2.2, Theorem 2.7 and Lemma 3.3, we can check the lowest $Sp(p) \times Sp(1)$ -types of I correspond to some lowest degree $U(2)$ -types of F_{λ_1, λ_2} in $\mathcal{H}_{p,1,2}$ as follows.

If $\lambda_1 - \lambda_2 \leq 2p - 1$, by Theorem 2.2 and Lemma 3.3, the unique lowest $Sp(p) \times Sp(1)$ -type of I is $(0, \dots, 0; \lambda_1 - \lambda_2 - 1)$, which corresponds to the $U(2)$ -type $(p-1, p - \lambda_1 + \lambda_2)$ in $\mathcal{H}_{p,1,2}$. Since $\lambda_1 \geq p$, $\lambda_1 + \lambda_2 \geq 1$ and $\lambda_1 - \lambda_2 \leq 2p - 1$, we check that $-\lambda_2 \leq p-1 \leq \lambda_1 - 1$. By Theorem 2.7 and Lemma 3.3, $(p-1, p - \lambda_1 + \lambda_2)$ is a lowest degree $U(2)$ -type of F_{λ_1, λ_2} .

If $\lambda_1 - \lambda_2 > 2p - 1$, by Theorem 2.2, the lowest $Sp(p) \times Sp(1)$ -types of I are

$$\left(\frac{\lambda_1 - \lambda_2 + 1 + x}{2} - p, 0, \dots, 0; \frac{\lambda_1 - \lambda_2 - 3 - x}{2} + p \right), \quad x = 0, \pm 1.$$

Here x is chosen so that $\lambda_1 - \lambda_2 + 1 + x$ is even. By Lemma 3.3, the lowest $Sp(p) \times Sp(1)$ -types correspond to

$$\left(\frac{\lambda_1 - \lambda_2 - 1 + x}{2}, \frac{1 + \lambda_2 - \lambda_1 + x}{2} \right) \quad (55)$$

in $\mathcal{H}_{p,1,2}$. Since $\lambda_1 - \lambda_2 > 2p - 1$, we know that $\frac{1+\lambda_2-\lambda_1+x}{2} \leq p - 1 \leq \frac{\lambda_1-\lambda_2-1+x}{2}$. By Lemma 3.3, the $U(2)$ -types in (55) are lowest degree $U(2)$ -types in I' of degree $\lambda_1 - \lambda_2 - 1$. Since $\lambda_1 + \lambda_2 \geq 1$, we check that $-\lambda_2 \leq \frac{\lambda_1-\lambda_2-1+x}{2} \leq \lambda_1 - 1$. By Theorem 2.7, the $U(2)$ -types in (55) are lowest degree $U(2)$ -types of F_{λ_1, λ_2} .

By Theorem 3.2, we know that $\theta_{p,1}(F_{\lambda_1, \lambda_2})$ is the constituent of I containing all lowest $Sp(p) \times Sp(1)$ -types. We finish the proof.

Next we calculate the explicit theta $(p, 1)$ -lifts of limits of highest weight discrete series $\overline{D}_{\lambda_1, \lambda_2}$.

Theorem 6.4. *The following is a list of explicit theta $(p, 1)$ -lifts of limits of highest weight discrete series for $p \geq 1$:*

(1) *If $\lambda_1 \geq p$, then*

$$\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2}) = \pi((\lambda_1, p - 1, p - 2, \dots, 1; -\lambda_2), \Psi_2).$$

(2) *If $\lambda_1 < p$, then*

$$\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2}) = 0.$$

PROOF. First assume $\lambda_1 < p$. By Theorem 2.7 and Lemma 3.3, every $U(2)$ -type in $\mathcal{K}(\overline{D}_{\lambda_1, \lambda_2})$ does not occur in the joint harmonics $\mathcal{H}_{p,1,2}$. Then $\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2})$ is zero by Theorem 3.2.

Next assume $\lambda_1 \geq p$. Let Φ be the nonzero projection from $\omega_{p,1,2}$ to $I(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2)) \otimes I'(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2)$ as in (54). Denote the first standard module by I and the second standard module by I' . By Theorem 2.5, $\overline{D}_{\lambda_1, \lambda_2}$ can be realized as a quotient of I' . By Theorem 2.7 and Lemma 3.3, we know that $\eta' = (\lambda_1 - 1, \lambda_2)$ is a lowest degree $U(2)$ -type of $\overline{D}_{\lambda_1, \lambda_2}$ of degree $\lambda_1 - \lambda_2 - 1$. As in the proof of Theorem 6.3, we prove that $\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2})$ is a constituent of I . It will be proven in Proposition 6.7 that $\eta = (\lambda_1 - p, 0, \dots, 0; p - 1 - \lambda_2)$ occurs in $\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2})$ as a lowest $Sp(p) \times Sp(1)$ -type. By Theorem 3.5, the infinitesimal character of $\overline{D}_{\lambda_1, \lambda_2}$ is $(\lambda_1, \lambda_2, p - 1, p - 2, \dots, 1)$. By the appendix, the unique element in $\mathcal{R}(Sp(p, 1))$ satisfying these conditions is $\pi((\lambda_1, p - 1, p - 2, \dots, 1; -\lambda_2), \Psi_2)$. We finish the proof.

Finally, we calculate the explicit theta $(p, 1)$ -lifts of limits of lowest weight discrete series D_{λ_1, λ_2} .

Theorem 6.5. *The following is a list of explicit theta $(p, 1)$ -lifts of limits of lowest weight discrete series for $p \geq 1$:*

(1) *If $\lambda_1 \leq p - 1$ and $\lambda_1 + \lambda_2 \neq 0$, then*

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi(1, (p - 1, p - 2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

(2) *If $\lambda_1 \leq p - 1$ and $\lambda_1 + \lambda_2 = 0$, then*

$$\theta_{p,1}(D_{\lambda_1, -\lambda_1}) = \pi((p - 1, \dots, \lambda_1 + 1, \lambda_1, \lambda_1, \lambda_1 - 1, \dots, 1; \lambda_1), \Psi)$$

with Ψ uniquely determined by the condition (A) of §2.1.

(3) If $\lambda_1 \geq p$ and $\lambda_2 \leq -p$, then

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, p-1, p-2, \dots, 1; -\lambda_2), \Psi_4).$$

(4) If $p \geq 2$, $\lambda_1 \geq p$ and $1-p < \lambda_2 \leq p-2$, then

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi(1, (\lambda_1, p-2, p-3, \dots, 1), \Psi_1, p-1-\lambda_2, p-1+\lambda_2).$$

(5) If $p \geq 2$, $\lambda_1 \geq p$ and $\lambda_2 = 1-p$, then

$$\theta_{p,1}(D_{\lambda_1, 1-p}) = \pi((\lambda_1, p-1, p-2, \dots, 1; p-1), \Psi_4).$$

(6) If $p \geq 3$ and $\lambda_2 \geq p-1$, then

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi(1, (\lambda_1, \lambda_2, p-3, p-4, \dots, 1), \Psi_1, 1, 2p-3).$$

(7) If $p = 2$ and $\lambda_2 \geq 1$, then

$$\theta_{2,1}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, \lambda_2; 1), \Psi_3).$$

(8) If $p = 1$ and $\lambda_2 \geq 0$, then

$$\theta_{1,1}(D_{\lambda_1, \lambda_2}) = 0.$$

PROOF. Let Φ be the nonzero projection from $\omega_{p,1,2}$ to $I(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \otimes I'(\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2))$ as in (54). Denote the first standard module by I and the second standard module by I' . By Theorem 2.5, D_{λ_1, λ_2} can be realized as a quotient of I' .

If $\lambda_1 \leq p-1$, by Theorem 2.7 and Lemma 3.3, we know that $\eta' = (p-1, p-\lambda_1+\lambda_2)$ is a lowest degree $U(2)$ -type of D_{λ_1, λ_2} of degree $\lambda_1 - \lambda_2 - 1$. As in the proof of Theorem 6.3, we prove that $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ is a constituent of I . If $\lambda_1 + \lambda_2 \neq 0$, by Theorem 2.2 and Lemma 3.3, we know that $\eta = (0, \dots, 0; \lambda_1 - \lambda_2 - 1)$ is a lowest $Sp(p) \times Sp(1)$ -type of I and corresponds to η' in $\mathcal{H}_{p,1,2}$. By Theorem 3.2, $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ is the unique constituent of I containing all lowest $Sp(p) \times Sp(1)$ -types. Then

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1), \Psi_1, \lambda_1 - \lambda_2, \lambda_1 + \lambda_2).$$

If $\lambda_1 + \lambda_2 = 0$, the standard module I is just the limit of discrete series (see Lemma 3.2.6 of [14])

$$\pi((p-1, \dots, \lambda_1+1, \lambda_1, \lambda_1-1, \dots, 1; \lambda_1), \Psi)$$

with Ψ uniquely determined by condition (A) since the Harish-Chandra parameter $(p-1, \dots, \lambda_1+1, \lambda_1, \lambda_1-1, \dots, 1; \lambda_1)$ satisfies condition (B).

If $\lambda_1 \geq p$ and $\lambda_2 \leq p-2$, by Theorem 2.7 and Lemma 3.3, we know that $\eta' = (\lambda_1, \lambda_2+1)$ is a lowest degree $U(2)$ -type of D_{λ_1, λ_2} of degree $\lambda_1 - \lambda_2 - 1$. As in the proof of Theorem 6.3, we prove that $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ is a constituent of I . It will be proven in Proposition 6.6 that $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ has a lowest $Sp(p) \times Sp(1)$ -type $(\lambda_1 - p + 1, 0, \dots, 0; p-2-\lambda_2)$. By Theorem 3.5, the infinitesimal character of $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ is $(\lambda_1, \lambda_2, p-1, p-2, \dots, 1)$. By the appendix, there is a unique element in $\mathcal{R}(Sp(p, 1))$ satisfying these conditions. The explicit theta $(p, 1)$ -lifts are listed above.

If $p \geq 3$ and $\lambda_2 \geq p-1$, by Theorem 6.1 and Theorem 4.6,

$$\theta_{p,1}(D_{\lambda_1, \lambda_2}) = \pi(1, (\lambda_1, \lambda_2, p-3, p-4, \dots, 1), \Psi_1, 1, 2p-3).$$

If $p = 1$ and $\lambda_2 \geq 0$, by Theorem 2.7 and Lemma 3.3, every $U(2)$ -type in $\mathcal{K}(D_{\lambda_1, \lambda_2})$ does not occur in the joint harmonics $\mathcal{H}_{1,1,2}$. Then

$$\theta_{1,1}(D_{\lambda_1, \lambda_2}) = 0.$$

If $p = 2$ and $\lambda_2 \geq 2$, by Proposition 6.1 and Theorem 6.2 of [10],

$$\theta_{2,1}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, \lambda_2; 1), \Psi_3).$$

It will be proven in Proposition 6.9 that $\theta_{2,1}(D_{\lambda_1, 1})$ is nonzero. By Theorem 2.7, Lemma 3.3, Theorem 3.5 and Theorem 3.8, $\theta_{2,1}(D_{\lambda_1, 1})$ has a unique lowest $Sp(2) \times Sp(1)$ -type $(\lambda_1 - 1, 1; 0)$ and its infinitesimal character is $(\lambda_1, 1, 1)$. It is not difficult to check the unique element in $\mathcal{R}(Sp(p, 1))$ satisfying these conditions is $\pi((\lambda_1, 1; 1), \Psi_3)$. We finish the proof.

Here we give the proofs of three conclusions which remain unproved until now.

Proposition 6.6. *If $\lambda_1 \geq p$, $\lambda_2 \leq p - 2$ and $\lambda_1 + \lambda_2 \geq 0$, then $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ has a unique lowest $Sp(p) \times Sp(1)$ -type $(\lambda_1 - p + 1, 0, \dots, 0; p - 2 - \lambda_2)$.*

PROOF. By Theorem 2.7 and Lemma 3.3, we know that

$$\mathcal{A}_{p,1,2}(D_{\lambda_1, \lambda_2}) = \{(k + \lambda_1 - \lambda_2 - 1, k) | \lambda_2 + 1 \leq k \leq p - 1\}.$$

By Theorem 3.2, Lemma 3.3 and Theorem 3.8, $\mathcal{A}(\theta_{p,1}(D_{\lambda_1, \lambda_2}))$ is contained in

$$\mathcal{D} = \{(k - p + \lambda_1 - \lambda_2, 0, \dots, 0; p - 1 - k) | \lambda_2 + 1 \leq k \leq p - 1\}.$$

Then the lowest $Sp(p) \times Sp(1)$ -types are those of minimal norm (see Definition 5.1 of [22]) in \mathcal{D} . Since $2\rho_c = (2p, 2(p - 1), \dots, 2; 2)$, we just need to find out the smallest integer of

$$\{(k + p + \lambda_1 - \lambda_2)^2 + (p + 1 - k)^2 | \lambda_2 + 1 \leq k \leq p - 1\}.$$

Since $\lambda_1 + \lambda_2 \geq 0$, the smallest integer is obtained when $k = \lambda_2 + 1$. This means that the unique lowest $Sp(p) \times Sp(1)$ -type of $\theta_{p,1}(D_{\lambda_1, \lambda_2})$ is $(\lambda_1 - p + 1, 0, \dots, 0; p - 2 - \lambda_2)$.

Proposition 6.7. *If $\lambda_1 \geq p$, $\lambda_2 \leq -p$ and $\lambda_1 + \lambda_2 \leq 0$, then $\theta_{p,1}(\overline{D}_{\lambda_1, \lambda_2})$ has a unique lowest $Sp(p) \times Sp(1)$ -type $(\lambda_1 - p, 0, \dots, 0; p - 1 - \lambda_2)$.*

PROOF. The proof is similar to the proof of Proposition 6.6.

To prove the occurrence of $D_{\lambda_1, 1}$ for $(Sp(2, 1), O^*(4))$, we need to introduce Witt Towers and conservation relations. We define an equivalence relation on the set of right \mathbb{H} Hermitian spaces. We say a right \mathbb{H} Hermitian space V_1 with signature (p_1, q_1) is equivalent to a right \mathbb{H} Hermitian space V_2 with signature (p_2, q_2) if and only if $p_1 - q_1 = p_2 - q_2$. A Witt Tower T is defined to be an equivalence class in the set of right \mathbb{H} Hermitian spaces. For non-negative integers p and q , let $\mathbb{H}^{p,q}$ be the right column vector space \mathbb{H}^{p+q} with Hermitian form (1). For $\pi' \in \mathcal{R}(O^*(4))$, we define

$$n_T(\pi') = \min\{p + q | \theta_{p,q}(\pi') \neq 0, \mathbb{H}^{p,q} \in T\}.$$

For two Witt towers T_1 and T_2 , we define the distance between T_1 and T_2 to be

$$\text{dist}(T_1, T_2) = |p_1 + q_2 - q_1 - p_2|$$

if $\mathbb{H}^{p_1, q_1} \in T_1$ and $\mathbb{H}^{p_2, q_2} \in T_2$. It is easy to check that the distance is well-defined. We have the following Theorem (see Theorem 7.6 of [21]).

Theorem 6.8. *For $\pi' \in \mathcal{R}(O^*(4))$, there are two different Witt towers T_1 and T_2 such that*

$$n_{T_1}(\pi') + n_{T_2}(\pi') = 5.$$

For two different Witt towers T_3 and T_4 ,

$$n_{T_3}(\pi') + n_{T_4}(\pi') \geq 4 + \text{dist}(T_3, T_4).$$

Proposition 6.9. *Let $\pi' = D_{\lambda_1, 1}$. Then $\theta_{2,1}(\pi')$ is nonzero.*

PROOF. We denote by T_1 the Witt tower containing $\mathbb{H}^{2,0}$ and by T_2 the Witt tower containing $\mathbb{H}^{2,1}$. By Theorem 5.1 of [11] and Theorem 6.1, $\theta_{p,q}(\pi') = 0$ for all $p + q \leq 2$ except $(2, 0)$. Then $n_T(\pi') \geq 2$ for any Witt tower T and the equality holds if and only if $T = T_1$. By Theorem 6.8, there exists a Witt tower T' such that $n_{T'}(\pi') = 3$. By Theorem 6.1 and Theorem 6.8, $\theta_{3,0}(\pi') = 0$ and T_2 is the unique Witt Tower T' such that $n_{T'}(\pi') = 3$. This means that $\theta_{2,1}(\pi')$ is nonzero.

6.3. Explicit Theta $(p, 2)$ -Lifting for $O^*(4)$

Assume $p \geq 2$. By results in §6.2, the following is a list of infinitesimal equivalence classes in $\mathcal{R}(O^*(4))$ whose theta $(p-1, 1)$ -lifts are zero:

- (1) Limits of highest weight discrete series $\overline{D}_{\lambda_1, \lambda_2}$, $\lambda_1 < p-1$;
- (2) Irreducible finite dimensional representations F_{λ_1, λ_2} , $\lambda_1 < p-1$;
- (3) $p = 2$, Lowest weight discrete series D_{λ_1, λ_2} , $\lambda_2 \geq 0$.

By Theorem 3.6 and Theorem 4.6, the problem is reduced to calculating the theta $(p, 2)$ -lifts of the elements listed above. We denote by $\mathbf{G}_{p,2}$ the set $\{\overline{D}_{\lambda_1, \lambda_2}, F_{\lambda_1, \lambda_2} | \lambda_1 < p-1\}$. If $\lambda_1 < p-1$, by Theorem 2.7 and Lemma 3.3, the unique lowest degree $U(2)$ -type of $\overline{D}_{\lambda_1, \lambda_2}$ (resp. F_{λ_1, λ_2}) is $(\lambda_1 - 1, \lambda_2)$. By Theorem 3.3, Lemma 3.3 and Theorem 3.8, we know that the unique lowest $Sp(p) \times Sp(2)$ -type of $\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2})$ (resp. $\theta_{p,2}(F_{\lambda_1, \lambda_2})$) is of the form $(0, \dots, 0; b_1, b_2)$. We denote by $\theta_{p,2}(\mathbf{G}_{p,2})$ the set of theta $(p, 2)$ -lifts of elements in $\mathbf{G}_{p,2}$ and denote by $\mathbf{B}_{p,2}$ the set of infinitesimal equivalence classes in $\mathcal{R}(Sp(p, 2))$ whose lowest $Sp(p) \times Sp(2)$ -types are all of the form $(0, \dots, 0; b_1, b_2)$. By the appendix, we have the following theorem.

Theorem 6.10. *The following is a list of infinitesimal equivalence classes in $\mathbf{B}_{p,2}$:*

- (1) $\pi((p, p-1, \dots, 1; \delta_1, \delta_2), \Psi_5)$, $\delta_1 > \delta_2 \geq p$;
- (2) $\pi((p-1, \dots, \delta_2+1, \delta_2, \delta_2-1, \dots, 1; \delta_1, \delta_2), \Psi)$, $\delta_1 \geq p-1 \geq \delta_2$. Here Ψ is the unique system of positive roots such that condition (A) is satisfied and $-e_1 + f_1, e_1 - f_2 \in \Psi$;
- (3) $\pi((\lambda_1, \lambda_2, \dots, \lambda_p; \delta_1, \delta_2), \Psi)$, $\delta_1 \leq \lambda_1 = p-2$. Here Ψ is the unique system of positive roots such that condition (A) is satisfied;
- (4) $\pi(1, (p-1, p-2, \dots, 1; \delta), \Psi_2, \mu, \nu)$, $\delta \geq p-1$. Here $\mu \leq 2p-1$ and the equality holds only if $\delta > p-1$;
- (5) $\pi(1, (p-2, \dots, \delta+1, \delta, \delta, \delta-1, \dots, 1; \delta), \Psi, \mu, \nu)$, $\delta \leq p-2$, $\mu \leq 2p-3$. Here Ψ is uniquely determined by condition (A);
- (6) $\pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (\nu_1, \nu_2))$, $\mu_1 \geq \mu_2$ and $\mu_1 \leq 2p-3$.

We denote by $\mathbf{B}_{p,2}^{(1)}$ the set of infinitesimal equivalence classes in Case (1) of Theorem 6.10. Similarly, we define the sets $\mathbf{B}_{p,2}^{(2)}$, $\mathbf{B}_{p,2}^{(3)}$, $\mathbf{B}_{p,2}^{(4)}$, $\mathbf{B}_{p,2}^{(5)}$ and $\mathbf{B}_{p,2}^{(6)}$. Denote $\mathbf{B}_{p,2}^+ = \mathbf{B}_{p,2}^{(1)} \cup \mathbf{B}_{p,2}^{(4)} \cup \mathbf{B}_{p,2}^{(6)}$ and $\mathbf{B}_{p,2}^- = \mathbf{B}_{p,2}^{(2)} \cup \mathbf{B}_{p,2}^{(3)} \cup \mathbf{B}_{p,2}^{(5)}$. We have the following theorem.

Theorem 6.11. *The set $\theta_{p,2}(\mathbf{G}_{p,2})$ is a subset of $\mathbf{B}_{p,2}^+$.*

PROOF. Since $\theta_{p,2}(\mathbf{G}_{p,2})$ is a subset of $\mathbf{B}_{p,2}$ and $\mathbf{B}_{p,2} = \mathbf{B}_{p,2}^+ \cup \mathbf{B}_{p,2}^-$, we only need to prove that $\theta_{p,2}(\mathbf{G}_{p,2}) \cap \mathbf{B}_{p,2}^- = \emptyset$.

Suppose π is an element in $\mathbf{B}_{p,2}^{(3)}$. The absolute values of all entries in the infinitesimal character of π are less than p . On the other hand, by Theorem 3.5, the infinitesimal character of $\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2})$ (resp. $\theta_{p,2}(F_{\lambda_1, \lambda_2})$) is $(\lambda_1, \lambda_2, p, p-1, \dots, 1)$. Then π is not the theta $(p, 2)$ -lift of an element in $\mathbf{G}_{p,2}$. Then we prove that $\theta_{p,2}(\mathbf{G}_{p,2}) \cap \mathbf{B}_{p,2}^{(3)} = \emptyset$.

Suppose π is an element in $\mathbf{B}_{p,2}^{(2)}$ as in Theorem 6.10. By Theorem 3.5, the infinitesimal character of π corresponds to the infinitesimal character of an element in $\mathbf{G}_{p,2}$ only if $\delta_1 = p$. If $\delta_1 = p$, the unique element in $\mathbf{G}_{p,2}$ whose infinitesimal character corresponds to the infinitesimal character of π is $\overline{D}_{\delta_2, -\delta_2}$. By Theorem 2.2, the lowest $Sp(p) \times Sp(2)$ -type of π is $(0, \dots, 0; 2p-2, 2\delta_2-1)$ while the lowest degree $U(2)$ -type of $\overline{D}_{\delta_2, -\delta_2}$ is $(\delta_2-1, -\delta_2)$. By Lemma 3.3, the $Sp(p) \times Sp(2)$ -type $(0, \dots, 0; 2p-2, 2\delta_2-1)$ does not correspond to the $U(2)$ -type $(\delta_2-1, -\delta_2)$ in $\mathcal{H}_{p,2,2}$. By Theorem 3.2 and Theorem 3.8, π is not the theta $(p, 2)$ -lifts of $\overline{D}_{\delta_2, -\delta_2}$. We prove that $\theta_{p,2}(\mathbf{G}_{p,2}) \cap \mathbf{B}_{p,2}^{(2)} = \emptyset$.

Suppose π is an element in $\mathbf{B}_{p,2}^{(5)}$ as in Theorem 6.10. By Theorem 3.5, the infinitesimal character of π corresponds to the infinitesimal character of an element in $\mathbf{G}_{p,2}$ only if $(\mu, \nu) = (1, 2p-1)$. If $(\mu, \nu) = (1, 2p-1)$, the unique element in $\mathbf{G}_{p,2}$ whose infinitesimal character corresponds to the infinitesimal character of π is $\overline{D}_{\delta, -\delta}$. By Theorem 2.2, the lowest $Sp(p) \times Sp(2)$ -type of π is $(0, \dots, 0; 2\delta-1, 0)$ while the lowest degree $U(2)$ -type of $\overline{D}_{\delta, -\delta}$ is $(\delta-1, -\delta)$. By Lemma 3.3, the $Sp(p) \times Sp(2)$ -type $(0, \dots, 0; 2\delta-1, 0)$ does not correspond to the $U(2)$ -type $(\delta-1, -\delta)$ in $\mathcal{H}_{p,2,2}$. By Theorem 3.2 and Theorem 3.8, π is not the theta $(p, 2)$ -lifts of $\overline{D}_{\delta, -\delta}$. Then we prove that $\theta_{p,2}(\mathbf{G}_{p,2}) \cap \mathbf{B}_{p,2}^{(5)} = \emptyset$.

In summary, we prove that $\theta_{p,2}(\mathbf{G}_{p,2}) \cap \mathbf{B}_{p,2}^- = \emptyset$.

Theorem 6.12. *The following is a list of theta $(p, 2)$ -lifts of elements in $\mathbf{G}_{p,2}$:*

(1) *If $\lambda_1, \lambda_2 \leq -p$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi((p, p-1, \dots, 1; -\lambda_2, -\lambda_1), \Psi_5).$$

(2) *If $\lambda_2 \leq -p+1 \leq \lambda_1$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi(1, (p-1, p-2, \dots, 1; -\lambda_2), \Psi_2, p-\lambda_1, p+\lambda_1).$$

(3) *If $\lambda_2 \geq -p+2$, then*

$$\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2}) = \pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (\nu_1, \nu_2))$$

with $(\mu_1, \mu_2) = (p-1-\lambda_2, p-\lambda_1)$ and $(\nu_1, \nu_2) = (p-1+\lambda_2, p+\lambda_1)$.

(4) *If $\lambda_1 < p-1$, then*

$$\theta_{p,2}(F_{\lambda_1, \lambda_2}) = \pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (\nu_1, \nu_2))$$

with $(\mu_1, \mu_2) = (p-1-\lambda_2, p-\lambda_1)$ and $(\nu_1, \nu_2) = (p-1+\lambda_2, p+\lambda_1)$.

PROOF. By Theorem 2.7, Lemma 3.3, Theorem 3.5 and Theorem 3.8, we know that $\theta_{p,2}(\overline{D}_{\lambda_1, \lambda_2})$ (resp. $\theta_{p,2}(F_{\lambda_1, \lambda_2})$) has a unique lowest $Sp(p) \times Sp(2)$ -type $(0, \dots, 0; p-2-\lambda_2, p-1-\lambda_1)$ and its infinitesimal character is $(\lambda_1, \lambda_2, p, p-1, \dots, 1)$. By Theorem 6.11, there is a unique element in $\mathbf{B}_{p,2}^+$ satisfying these condition. We list the explicit theta $(p, 2)$ -lifts above.

We denote by $\mathbf{S}_{2,2}$ the set $\{D_{\lambda_1, \lambda_2} | \lambda_2 \geq 0\}$, and by $\mathbf{C}_{2,2}$ the set of elements in $\mathcal{R}(Sp(2, 2))$ whose lowest $Sp(2) \times Sp(2)$ -types are all of the form $(a_1, a_2; 0, 0)$. In the appendix, we list all infinitesimal equivalence classes in $\mathbf{C}_{2,2}$ in terms of Langlands parameters.

Theorem 6.13. *The following is a list of explicit theta $(2, 2)$ -lifts of elements in $\mathbf{S}_{2,2}$:*

(1) If $\lambda_2 = 0$, then

$$\theta_{2,2}(D_{\lambda_1, 0}) = \pi(1, (\lambda_1; 1), \Psi_3, 2, 2).$$

(2) If $\lambda_2 = 1$, then

$$\theta_{2,2}(D_{\lambda_1, 1}) = \pi(1, (\lambda_1; 1), \Psi_3, 3, 1).$$

(3) If $\lambda_2 \geq 2$, then

$$\theta_{2,2}(D_{\lambda_1, \lambda_2}) = \pi((\lambda_1, \lambda_2; 2, 1), \Psi_6).$$

PROOF. By Theorem 2.7, Lemma 3.3, Theorem 3.5 and Theorem 3.8, we know that $\theta_{2,2}(D_{\lambda_1, \lambda_2})$ has a unique lowest $Sp(2) \times Sp(2)$ -type $(\lambda_1, \lambda_2 + 1; 0, 0)$ and its infinitesimal character is $(\lambda_1, \lambda_2, 2, 1)$. By the appendix, there is a unique element in $\mathbf{C}_{2,2}$ satisfying these conditions. We list the explicit theta $(2, 2)$ -lifts above.

Appendix

In this appendix, we list infinitesimal equivalence classes in $\mathbf{A}_{p,1}$, $\mathbf{B}_{p,2}$ and $\mathbf{C}_{2,2}$ in terms of Langlands parameters for $p \geq 2$. Let $\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5$ and Ψ_6 be those mentioned in §5 and §6. Let condition (A) and (B) be those of §2.1.

First, we list infinitesimal equivalence classes in $\mathbf{A}_{p,1}$ in terms of Langlands parameters. It is obvious that $\mathbf{A}_{1,1} = \mathcal{R}(Sp(1, 1))$. Assume $p \geq 2$. By Theorem 2.2, the following is a list of infinitesimal equivalence classes in $\mathbf{A}_{p,1}$ for $p \geq 2$:

- (1) $\pi((\lambda_1, p-1, p-2, \dots, 1; \delta_1), \Psi_2)$, $\delta_1 \geq \lambda_1 > p-1$;
- (2) $\pi((\lambda_1, p-1, p-2, \dots, 1; \delta_1), \Psi_4)$, $\lambda_1 \geq \delta_1 > p-1$;
- (3) $\pi((\lambda_1, p-1, p-2, \dots, 1; p-1), \Psi_4)$, $\lambda_1 \geq p-1$;
- (4) $\pi((\lambda_1, p-2, \dots, \delta_1+1, \delta_1, \delta_1-1, \dots, 1; \delta_1), \Psi)$, $\delta_1 \leq p-2$. Here Ψ is uniquely determined by condition (A);
- (5) $\pi(1, (p-1, p-2, \dots, 1), \Psi_1, \mu, \nu)$;
- (6) $\pi(1, (\lambda_1, p-2, p-3, \dots, 1), \Psi_1, \mu, \nu)$, $\lambda_1 > p-1$ and $\mu \leq 2p-3$.

The desired theta $(p, 1)$ -lifts in Theorem 5.2, Theorem 6.4 and Theorem 6.5 are elements in $\mathbf{A}_{p,1}$. We determined these theta $(p, 1)$ -lifts in terms of Langlands parameters by their lowest $Sp(p) \times Sp(1)$ -types and infinitesimal characters.

Next we list infinitesimal equivalence classes in $\mathbf{B}_{p,2}$ in terms of Langlands parameters. Assume $p \geq 2$. For $\pi \in \mathcal{R}(Sp(p, 2))$, we denote by $\mathcal{A}^1(\pi)$ the set $\{a_1 | (a_1, a_2, \dots, a_p; b_1, b_2) \in \mathcal{A}(\pi)\}$. We know that $\pi \in \mathbf{B}_{p,2}$ if and only if $\mathcal{A}^1(\pi) = \{0\}$.

Suppose $\pi = \pi((\lambda_1, \lambda_2, \dots, \lambda_p; \delta_1, \delta_2), \Psi)$. By Theorem 2.2, the set $\mathcal{A}^1(\pi)$ is:

$$\left\{ \begin{array}{ll} \{\lambda_1 - p\} & \text{if } \lambda_1 < \delta_2, \\ \{\lambda_1 - p + 1\} & \text{if } \delta_2 < \lambda_1 < \delta_1, \\ \{\lambda_1 - p + 2\} & \text{if } \delta_1 < \lambda_1, \\ \{\lambda_1 - p\} & \text{if } \delta_1 > \delta_2 = \lambda_1 > \lambda_2 \text{ and } -e_1 + f_2 \in \Psi, \\ \{\lambda_1 - p + 1\} & \text{if } \delta_1 > \delta_2 = \lambda_1 > \lambda_2 \text{ and } e_1 - f_2 \in \Psi, \\ \{\lambda_1 - p + 1\} & \text{if } \delta_1 = \lambda_1 > \lambda_2, \delta_2 \text{ and } -e_1 + f_1 \in \Psi, \\ \{\lambda_1 - p + 2\} & \text{if } \delta_1 = \lambda_1 > \lambda_2, \delta_2 \text{ and } e_1 - f_1 \in \Psi, \\ \{\lambda_1 - p + 2\} & \text{if } \delta_1 = \lambda_1 = \lambda_2 > \delta_2, \\ \{\lambda_1 - p + 1\} & \text{if } \delta_2 = \lambda_1 = \lambda_2 < \delta_1, \\ \{\lambda_1 - p + 1\} & \text{if } \lambda_1 = \delta_1 = \delta_2 > \lambda_2, \\ \{\lambda_1 - p + 2\} & \text{if } \lambda_1 = \lambda_2 = \delta_1 = \delta_2 \text{ and } e_1 - f_1 \in \Psi, \\ \{\lambda_1 - p + 1\} & \text{if } \lambda_1 = \lambda_2 = \delta_1 = \delta_2 \text{ and } -e_1 + f_1 \in \Psi. \end{array} \right.$$

Suppose $\pi = \pi(1, (\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \delta_1), \Psi, \mu_1, \nu_1)$. By Theorem 2.2, the set $\mathcal{A}^1(\pi)$ is:

$$\left\{ \begin{array}{ll} \{\lambda_1 - p + 2\} & \text{if } \lambda_1 > \delta_1 \text{ and } \lambda_1 > \frac{\mu_1}{2}, \\ \{\lambda_1 - p + 1\}, & \text{if } \frac{\mu_1}{2} < \lambda_1 < \delta_1, \\ \{\frac{\mu_1+3}{2} - p\} & \text{if } \delta_1, \lambda_1 < \frac{\mu_1}{2} \text{ and } \mu_1 \text{ is odd,} \\ \{\frac{\mu_1+1}{2} - p\} & \text{if } \delta_1 > \frac{\mu_1}{2} > \lambda_1 \text{ and } \mu_1 \text{ is odd,} \\ \{\frac{\mu_1}{2} - p + 1, \frac{\mu_1}{2} - p + 2\} & \text{if } \delta_1, \lambda_1 < \frac{\mu_1}{2} \text{ and } \mu_1 \text{ is even,} \\ \{\frac{\mu_1}{2} - p, \frac{\mu_1}{2} - p + 1\} & \text{if } \delta_1 > \frac{\mu_1}{2} > \lambda_1 \text{ and } \mu_1 \text{ is even,} \\ \{\lambda_1 - p + 2\} & \text{if } \lambda_1 = \delta_1 \geq \frac{\mu_1}{2} \text{ and } e_1 - f_1 \in \Psi, \\ \{\lambda_1 - p + 1\} & \text{if } \lambda_1 = \delta_1 \geq \frac{\mu_1}{2} \text{ and } -e_1 + f_1 \in \Psi, \\ \{\lambda_1 - p + 2\} & \text{if } \lambda_1 = \frac{\mu_1}{2} > \delta_1, \\ \{\lambda_1 - p + 1\} & \text{if } \lambda_1 = \frac{\mu_1}{2} < \delta_1, \\ \{\frac{\mu_1}{2} - p + 1\} & \text{if } \delta_1 = \frac{\mu_1}{2} > \lambda_1. \end{array} \right.$$

Suppose $p \geq 3$ and $\pi = \pi(2, (\lambda_1, \lambda_2, \dots, \lambda_{p-2}), \Psi, (\mu_1, \mu_2), (\nu_1, \nu_2))$ with $\mu_1 \geq \mu_2$. By Theorem 2.2, the set $\mathcal{A}^1(\pi)$ is :

$$\left\{ \begin{array}{ll} \{\lambda_1 - p + 2\} & \text{if } \lambda_1 > \frac{\mu_1}{2}, \\ \{\frac{\mu_1+3}{2} - p\} & \text{if } \lambda_1 < \frac{\mu_1}{2} \text{ and } \mu_1 \text{ is odd,} \\ \{\frac{\mu_1}{2} - p + 1, \frac{\mu_1}{2} - p + 2\} & \text{if } \lambda_1 < \frac{\mu_1}{2} \text{ and } \mu_1 \text{ is even,} \\ \{\lambda_1 - p + 2\} & \text{if } \lambda_1 = \frac{\mu_1}{2}, \end{array} \right.$$

Suppose $p = 2$ and $\pi = \pi(2, 0, \emptyset, (\mu_1, \mu_2), (\nu_1, \nu_2))$ with $\mu_1 \geq \mu_2$. By Theorem 2.2, the set $\mathcal{A}^1(\pi)$ is:

$$\left\{ \begin{array}{ll} \{\frac{\mu_1-1}{2}\} & \text{if } \mu_1 \text{ is odd,} \\ \{\frac{\mu_1}{2} - 1, \frac{\mu_1}{2}\} & \text{if } \mu_1 \text{ is even.} \end{array} \right.$$

In summary, the following is a list of infinitesimal equivalence classes in $\mathbf{B}_{p,2}$:

- (1) $\pi((p, p-1, \dots, 1; \delta_1, \delta_2), \Psi_5), \delta_2 \geq p$;
- (2) $\pi((p-1, \dots, \delta_2+1, \delta_2, \delta_2, \delta_2-1, \dots, 1; \delta_1, \delta_2), \Psi), \delta_2 \leq p-1 \leq \delta_1$. Here Ψ is the unique system of positive roots such that condition (A) is satisfied and $-e_1 + f_1, e_1 - f_2 \in \Psi$;
- (3) $\pi((p-2, \lambda_2, \lambda_3, \dots, \lambda_p; \delta_1, \delta_2), \Psi), \delta_1 \leq p-2$. Here Ψ the unique system of positive roots such that condition (A) is satisfied;

- (4) $\pi(1, (p-1, p-2, \dots, 1; \delta_1), \Psi_2, \mu_1, \nu_1), \delta_1 > p-1$ and $\frac{\mu_1}{2} \leq p - \frac{1}{2}$;
- (5) $\pi(1, (p-1, p-2, \dots, 1; p-1), \Psi_2, \mu_1, \nu_1), \frac{\mu_1}{2} \leq p-1$;
- (6) $\pi(1, (p-2, \dots, \delta_1+1, \delta_1, \delta_1-1, \dots, 1; \delta_1), \Psi, \mu_1, \nu_1), \delta_1 \leq p-2, \frac{\mu_1}{2} \leq p - \frac{3}{2}$. Here Ψ is uniquely determined by condition (A);
- (7) $\pi(2, (p-2, p-3, \dots, 1), \Psi_1, (\mu_1, \mu_2), (\nu_1, \nu_2)), \mu_2 \leq \mu_1 \leq 2p-3$.

The infinitesimal equivalence classes listed above are just those listed in Theorem 6.10.

Finally, we list infinitesimal equivalence classes in $\mathbf{C}_{2,2}$ in terms of Langlands parameters. For $\pi \in \mathcal{R}(Sp(2, 2))$, we denote by $\mathcal{A}^3(\pi)$ the set $\{b_1 | (a_1, a_2; b_1, b_2) \in \mathcal{A}(\pi)\}$. We know that $\pi \in \mathbf{C}_{2,2}$ if and only if $\mathcal{A}^3(\pi) = \{0\}$. Let Ψ_7 be a system of positive roots for the group $Sp(2, 2)$ such that $(4, 2; 3, 1)$ is dominant with respect to Ψ_7 .

Suppose $\pi = \pi((\lambda_1, \lambda_2; \delta_1, \delta_2), \Psi)$. By Theorem 2.2, the set $\mathcal{A}^3(\pi)$ is

$$\left\{ \begin{array}{ll} \{\delta_1 - 2\}, & \text{if } \delta_1 < \lambda_2; \\ \{\delta_1 - 1\}, & \text{if } \lambda_2 < \delta_1 < \lambda_1; \\ \{\delta_1\}, & \text{if } \delta_1 > \lambda_1; \\ \{\delta_1 - 2\}, & \text{if } \lambda_1 > \lambda_2 = \delta_1 > \delta_2 \text{ and } e_2 - f_1 \in \Psi; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 > \lambda_2 = \delta_1 > \delta_2 \text{ and } -e_2 + f_1 \in \Psi; \\ \{\delta_1 - 1\}, & \text{if } \lambda_2, \delta_2 < \delta_1 = \lambda_1 \text{ and } e_1 - f_1 \in \Psi; \\ \{\delta_1\}, & \text{if } \lambda_2, \delta_2 < \delta_1 = \lambda_1 \text{ and } -e_1 + f_1 \in \Psi; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 = \lambda_2 = \delta_1 > \delta_2; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 > \lambda_2 = \delta_1 = \delta_2; \\ \{\delta_1\}, & \text{if } \lambda_1 = \delta_1 = \delta_2 > \lambda_2; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 = \lambda_2 = \delta_1 = \delta_2 \text{ and } e_1 - f_1 \in \Psi; \\ \{\delta_1\}, & \text{if } \lambda_1 = \lambda_2 = \delta_1 = \delta_2 \text{ and } -e_1 + f_1 \in \Psi. \end{array} \right.$$

Suppose $\pi = \pi(1, (\lambda_1; \delta_1), \Psi, \mu, \nu)$. By Theorem 2.2, the set $\mathcal{A}^3(\pi)$ is

$$\left\{ \begin{array}{ll} \{\frac{\mu-1}{2}\}, & \text{if } \lambda_1, \delta_1 < \frac{\mu}{2} \text{ and } \mu \text{ is odd;} \\ \{\frac{\mu}{2}, \frac{\mu}{2} - 1\}, & \text{if } \lambda_1, \delta_1 < \frac{\mu}{2} \text{ and } \mu \text{ is even;} \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 > \delta_1 > \frac{\mu}{2}; \\ \{\delta_1\}, & \text{if } \delta_1 > \lambda_1 \text{ and } \delta_1 \geq \frac{\mu}{2}; \\ \{\frac{\mu-3}{2}\}, & \text{if } \lambda_1 > \frac{\mu}{2} > \delta_1 \text{ and } \mu \text{ is odd;} \\ \{\frac{\mu}{2} - 1, \frac{\mu}{2} - 2\}, & \text{if } \lambda_1 > \frac{\mu}{2} > \delta_1 \text{ and } \mu \text{ is even;} \\ \{\frac{\mu}{2} - 1\}, & \text{if } \frac{\mu}{2} = \lambda_1 > \delta_1; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 > \delta_1 = \frac{\mu}{2}; \\ \{\delta_1 - 1\}, & \text{if } \lambda_1 = \delta_1 \geq \frac{\mu}{2} \text{ and } e_1 - f_1 \in \Psi; \\ \{\delta_1\}, & \text{if } \lambda_1 = \delta_1 \geq \frac{\mu}{2} \text{ and } -e_1 + f_1 \in \Psi. \end{array} \right.$$

Suppose $\pi = \pi(2, 0, \emptyset, (\mu_1, \mu_2), (\nu_1, \nu_2))$ with $\mu_1 \geq \mu_2$. By Theorem 2.2, the set $\mathcal{A}^3(\pi)$ is

$$\left\{ \begin{array}{ll} \{\frac{\mu_1-1}{2}\}, & \text{if } \mu_1 \text{ is odd;} \\ \{\frac{\mu_1}{2}, \frac{\mu_1}{2} - 1\}, & \text{if } \mu_1 \text{ is even.} \end{array} \right.$$

In summary, the following is a list of infinitesimal equivalence classes in $\mathbf{C}_{2,2}$:

- (1) $\pi((\lambda_1, \lambda_2; 2, 1), \Psi_6)$;
- (2) $\pi((\lambda_1, 1; 1, 1), \Psi_7)$;
- (3) $\pi(1, (\lambda_1; 1), \Psi_3, 3, \nu), \lambda_1 > 1$;
- (4) $\pi(1, (\lambda_1; 1), \Psi_3, \mu, \nu), \mu \leq 2$;
- (5) $\pi(2, 0, \emptyset, (1, 1), (\nu_1, \nu_2))$.

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References

- [1] J. Adams, D. Barbasch, Reductive dual pair correspondence for complex groups, *J. Funct. Anal.* 132 (1995) 1–42.
- [2] P. Cartier, Quantum mechanical commutation relations and theta functions, in: *Algebraic Groups and Discontinuous Subgroups* (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 361–383.
- [3] R. Howe, Transcending classical invariant theory, *J. Amer. Math. Soc.* 2 (1989) 535–552.
- [4] A.W. Knap, Representation theory of semisimple groups, *Princeton Landmarks in Mathematics*, Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.
- [5] A.W. Knap, D.A. Vogan, Jr., Cohomological induction and unitary representations, volume 45 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, 1995.
- [6] S.S. Kudla, On the local theta-correspondence, *Invent. Math.* 83 (1986) 229–255.
- [7] S. Lang, $SL_2(\mathbb{R})$, volume 105 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1986.
- [8] J.S. Li, On the classification of irreducible low rank unitary representations of classical groups, *Compositio Math.* 71 (1989) 29–48.
- [9] J.S. Li, Singular unitary representations of classical groups, *Invent. Math.* 97 (1989) 237–255.
- [10] J.S. Li, Theta lifting for unitary representations with nonzero cohomology, *Duke Math. J.* 61 (1990) 913–937.
- [11] J.S. Li, A. Paul, E.C. Tan, C.B. Zhu, The explicit duality correspondence of $(Sp(p, q), O^*(2n))$, *J. Funct. Anal.* 200 (2003) 71–100.
- [12] J.S. Li, E.C. Tan, C.B. Zhu, Tensor product of degenerate principal series and local theta correspondence, *J. Funct. Anal.* 186 (2001) 381–431.
- [13] C. Mœglin, Correspondance de Howe pour les paires reductives duales: quelques calculs dans le cas archimédien, *J. Funct. Anal.* 85 (1989) 1–85.
- [14] A. Paul, Howe correspondence for real unitary groups, *J. Funct. Anal.* 159 (1998) 384–431.
- [15] A. Paul, On the Howe correspondence for symplectic-orthogonal dual pairs, *J. Funct. Anal.* 228 (2005) 270–310.
- [16] V. Protsak, T. Przebinda, On the occurrence of admissible representations in the real Howe correspondence in stable range, *Manuscripta Math.* 126 (2008) 135–141.
- [17] T. Przebinda, The duality correspondence of infinitesimal characters, *Colloq. Math.* 70 (1996) 93–102.
- [18] R. Ranga Rao, On some explicit formulas in the theory of Weil representation, *Pacific J. Math.* 157 (1993) 335–371.
- [19] D. Shale, Linear symmetries of free boson fields, *Trans. Amer. Math. Soc.* 103 (1962) 149–167.
- [20] B. Speh, D.A. Vogan, Jr., Reducibility of generalized principal series representations, *Acta Math.* 145 (1980) 227–299.
- [21] B. Sun, C.B. Zhu, Conservation relations for local theta correspondence, *J. Amer. Math. Soc.*, to appear (2014).
- [22] D.A. Vogan, Jr., The algebraic structure of the representation of semisimple Lie groups. I, *Ann. of Math. (2)* 109 (1979) 1–60.
- [23] D.A. Vogan, Jr., Representations of real reductive Lie groups, volume 15 of *Progress in Mathematics*, Birkhäuser, Boston, Mass., 1981.
- [24] D.A. Vogan, Jr., Unitarizability of certain series of representations, *Ann. of Math. (2)* 120 (1984) 141–187.
- [25] D.A. Vogan, Jr., G.J. Zuckerman, Unitary representations with nonzero cohomology, *Compositio Math.* 53 (1984) 51–90.
- [26] N.R. Wallach, Real reductive groups. I, volume 132 of *Pure and Applied Mathematics*, Academic Press, Inc., Boston, MA, 1988.
- [27] A. Weil, Sur certains groupes d’opérateurs unitaires, *Acta Math.* 111 (1964) 143–211.